Math 300 Writing for Mathematics Spring 2003

Mathematics for Essay 2

Consider the infinite series:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots + \tag{1}$$

This is the case, $p = 2$ of the p-series:

$$\Sigma_{n=1}^{\infty} \frac{1}{n^p}.$$

We know from our discussion in Essay 1 that

$$\Sigma_{n=1}^{\infty} \frac{1}{n^2} < 2.$$

We noticed in Essay 1 that these are the areas of a collection of squares that fit in a 1 by 2 rectangle with a lot left over. We would like to get a better approximation to the actual value.

Sketch the graph of the function $f(x) = \frac{1}{x^2}$. Note that there are disjoint rectangles with area $\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$ contained in the region bounded by the x-axis, $f(x) = \frac{1}{x^2}$, and the line $x = 1$. (Take the rectangle formed by going one unit to the left from the point $(n, \frac{1}{n^2})$ to get area $\frac{1}{n^2}$.) So

$$\int_1^{\infty} \frac{1}{x^2}\,dx > \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots.$$

More generally

$$\int_m^{\infty} \frac{1}{x^2}\,dx > \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \frac{1}{(m+3)^2} + \cdots.$$

So for each m, we can get an upper bound on

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \tag{2}$$

as $1 + \frac{1}{4} + \frac{1}{9} + \cdots \frac{1}{m^2} + \int_m^{\infty} \frac{1}{x^2}\,dx$.

The first of these term is finite; find it with a calculator. On the other hand

$$\int_m^{\infty} \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_m^{\infty} = \lim_{b\to\infty} \left.\frac{-1}{x}\right|_m^{b} = \lim_{b\to\infty} (\frac{-1}{b} - \frac{-1}{m}) = \frac{1}{m}.$$

For example, taking $m = 2$, my estimate is $(1+1/4)+1/2 = 1\,3/4$, for $m = 3$ it is $(1 + 1/4 + 1/9) + 1/3$ which is approximately 1.69.

Now to estimate the volumes repeat this process but with $g(x) = \frac{1}{x^3}$ replacing $f(x)$. Remarkably, the exact value for $\Sigma_{n=1}^{\infty} \frac{1}{n^2}$ is $\frac{\pi^2}{6}$ while the exact value of $\Sigma_{n=1}^{\infty} \frac{1}{n^3}$ is unknown:
See www.mat.bham.ac.uk/C.J.Sangwin/Teaching/pus/infsersup.pdf