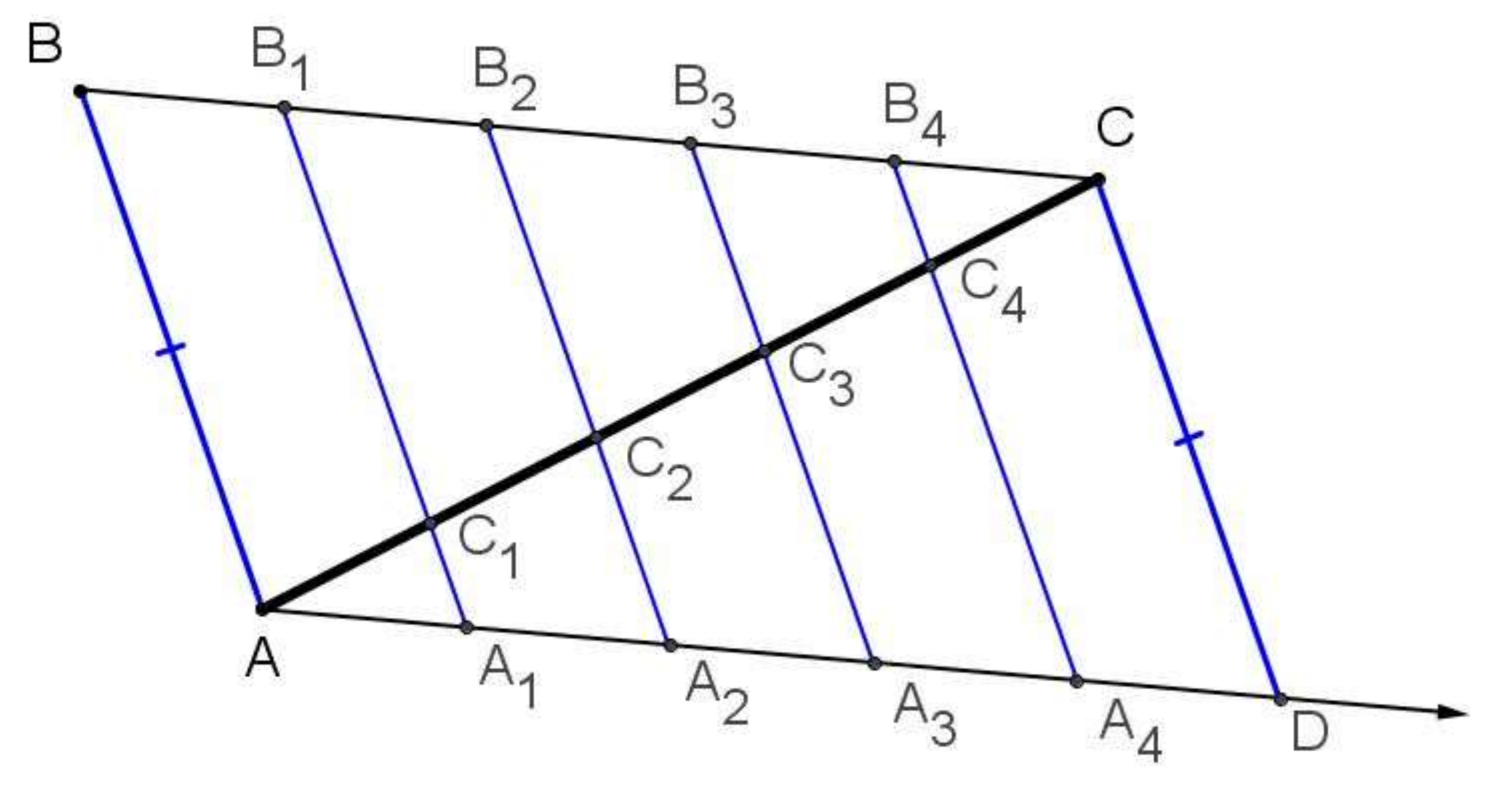
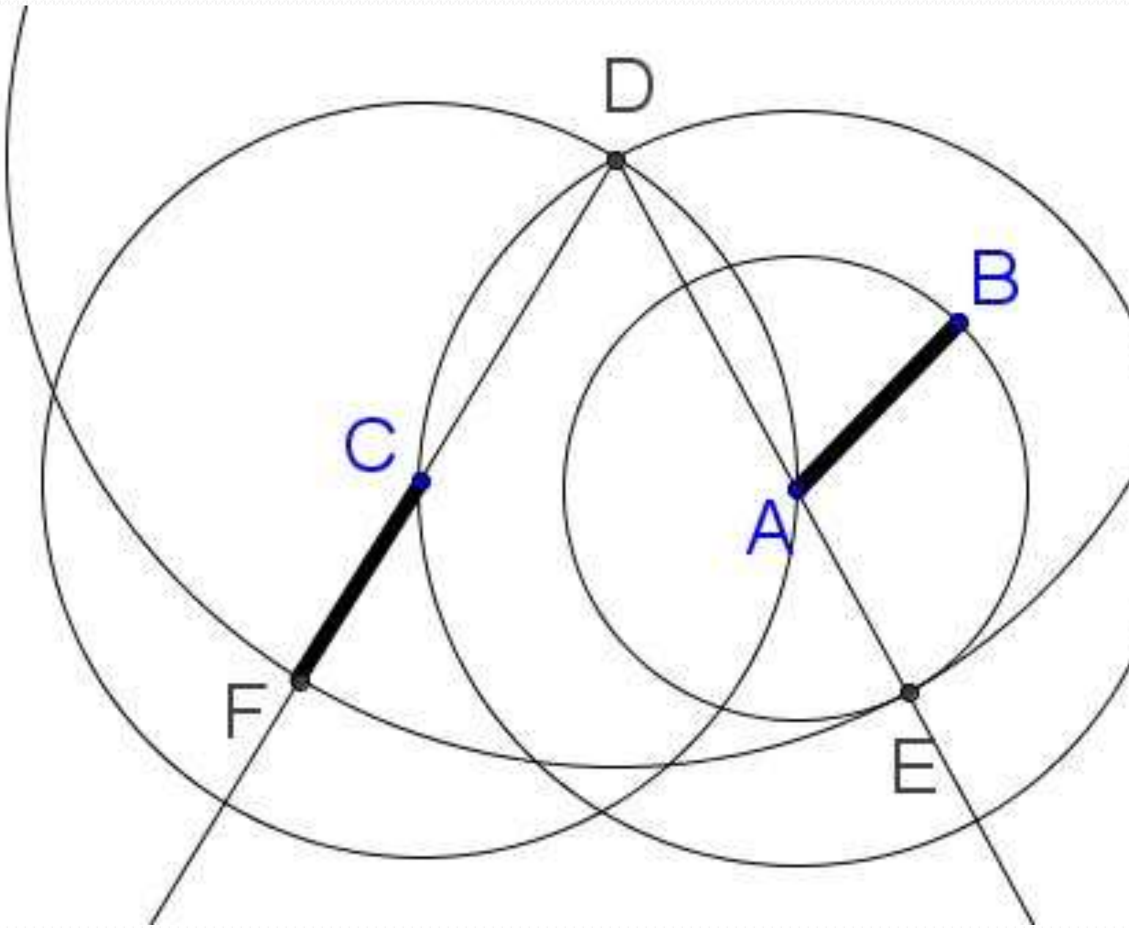
Initial Motivation



Rusty compass – prove it

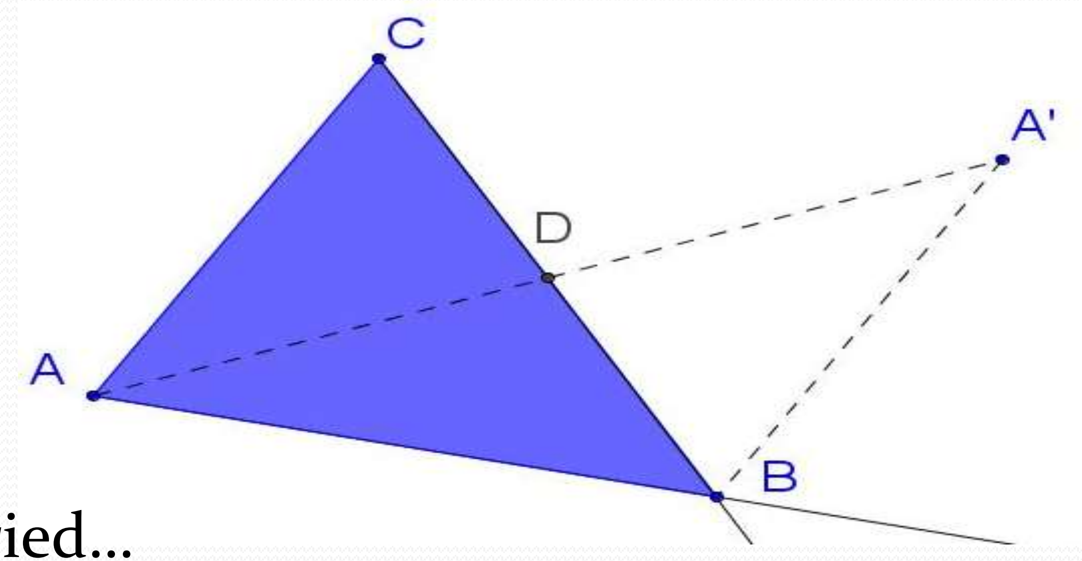


Circle through A with center C
Circle through C with center A
Intersection point of c, d
Ray through D, A
Ray through D, C
Circle through B with center A
Intersection point of f, b
Circle through E with center D
Intersection point of g, e
Segment [F, C]

Parallel Postulate

- 1. Existence

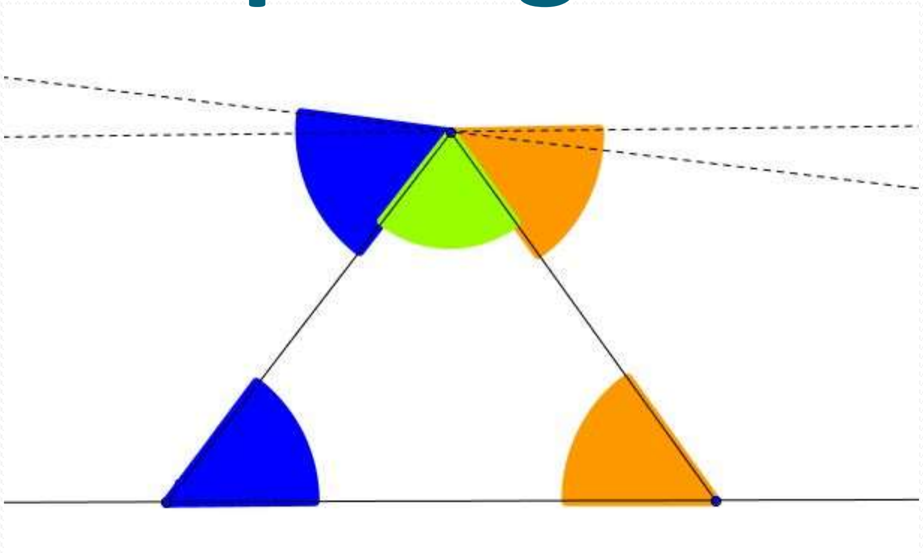
  Proof using the Weak Exterior Angle Theorem(Euclid I Proposition 16)



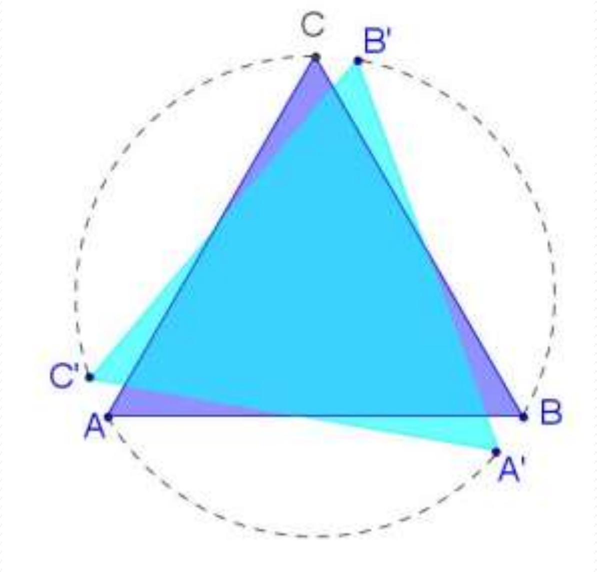
- 2. Uniqueness: we tried…

Equilateral to Equiangular

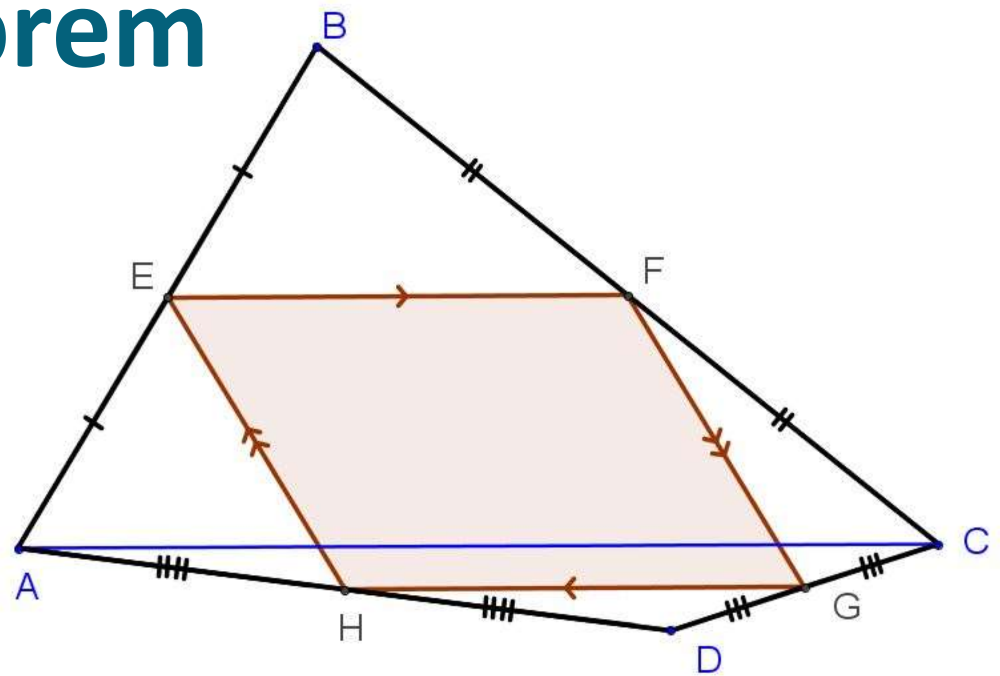
- Proving 60° w/o the //-Postulate
  - Teachers struggled
  - Great experience: why doesn't it work?

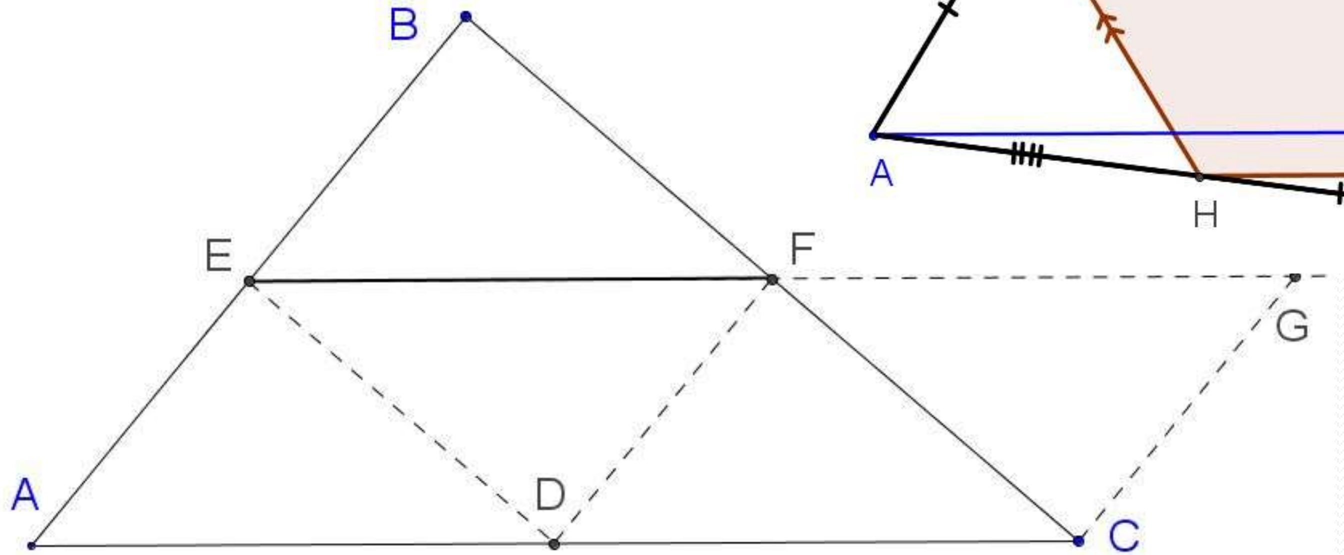


- Proving equiangular:
  - Terribly easy with Transformations (rotation or reflection)



Varignon's Theorem
(1731)



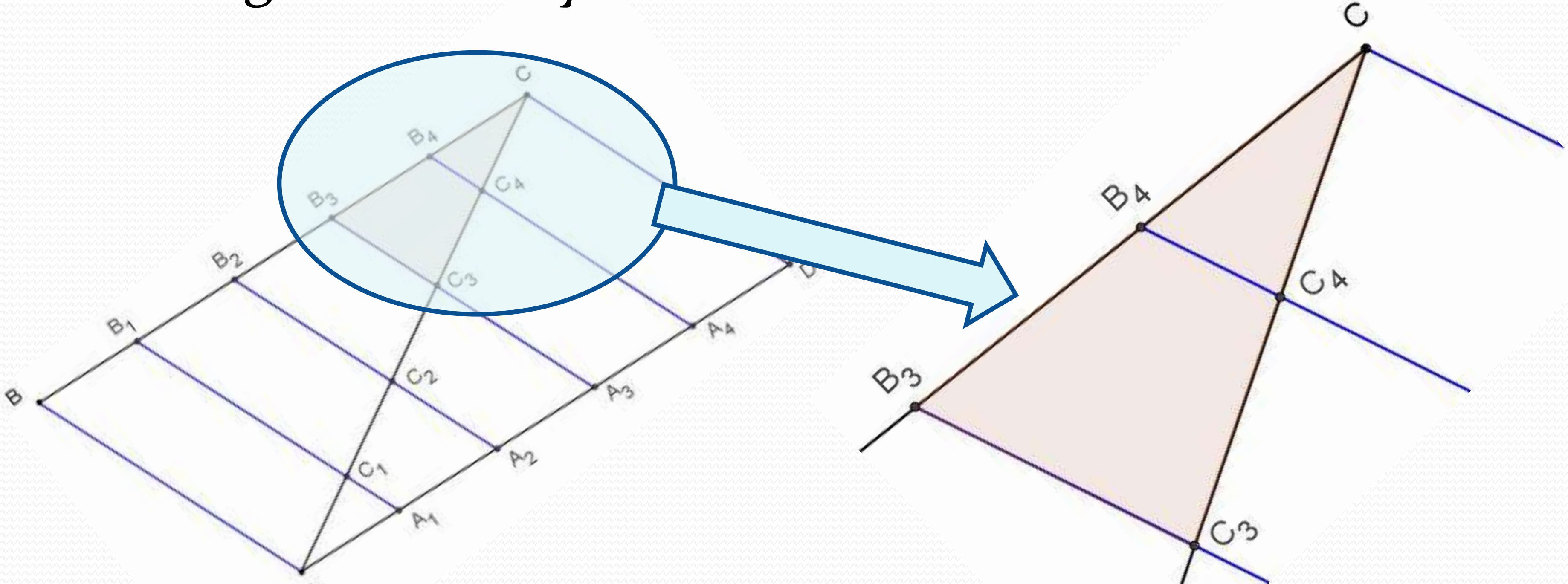


- Various teacher proofs

Rational Side Splitter

We're Done

- Varignon Activity Leads to Proof



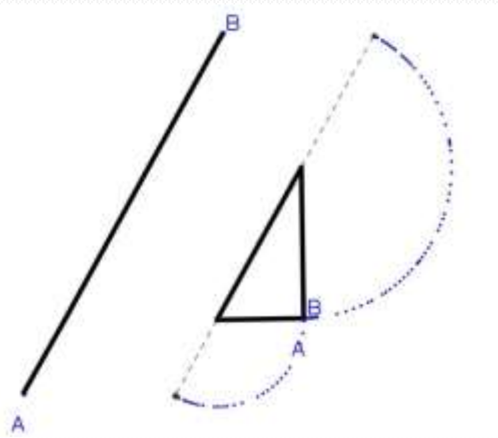
- Easy to generalize to rational splits
- Tempting to generalize for irrational numbers

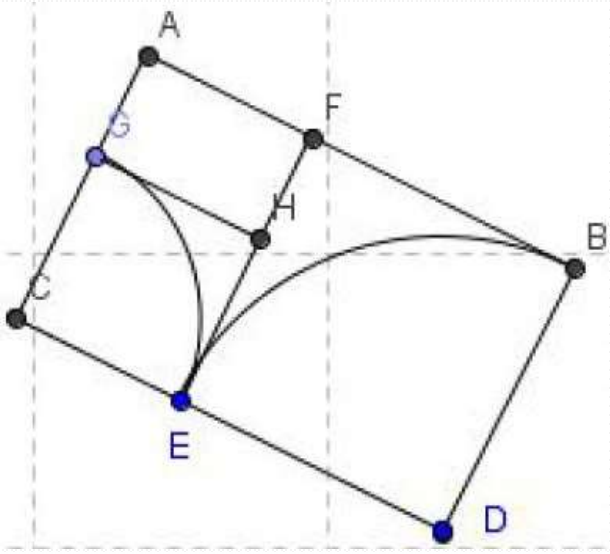
Needed: Motivation for Irrational Side Splitter

- Cut a given segment to make a 30-60-90-triangle

(306090.ggb)

- Cut a segment to make a Golden Rectangle

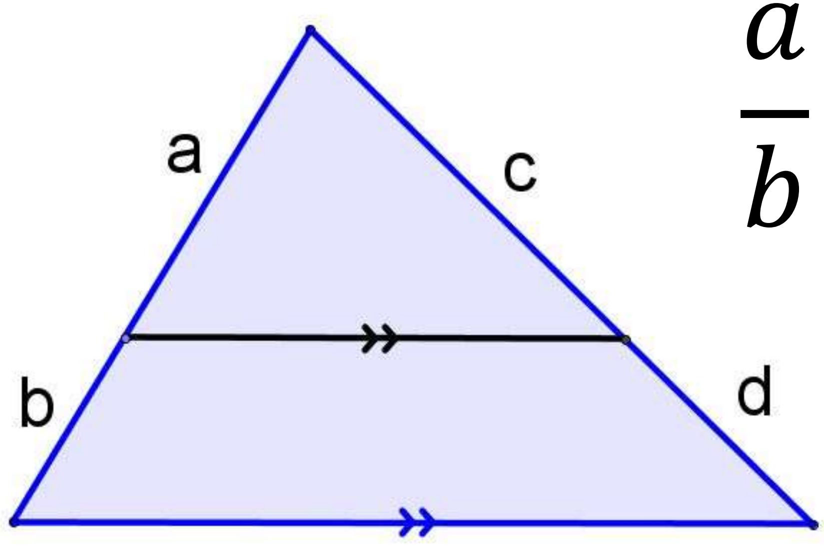




We proved that the Golden Ratio is irrational using Geometry

CME's Proof of Side-Splitter uses

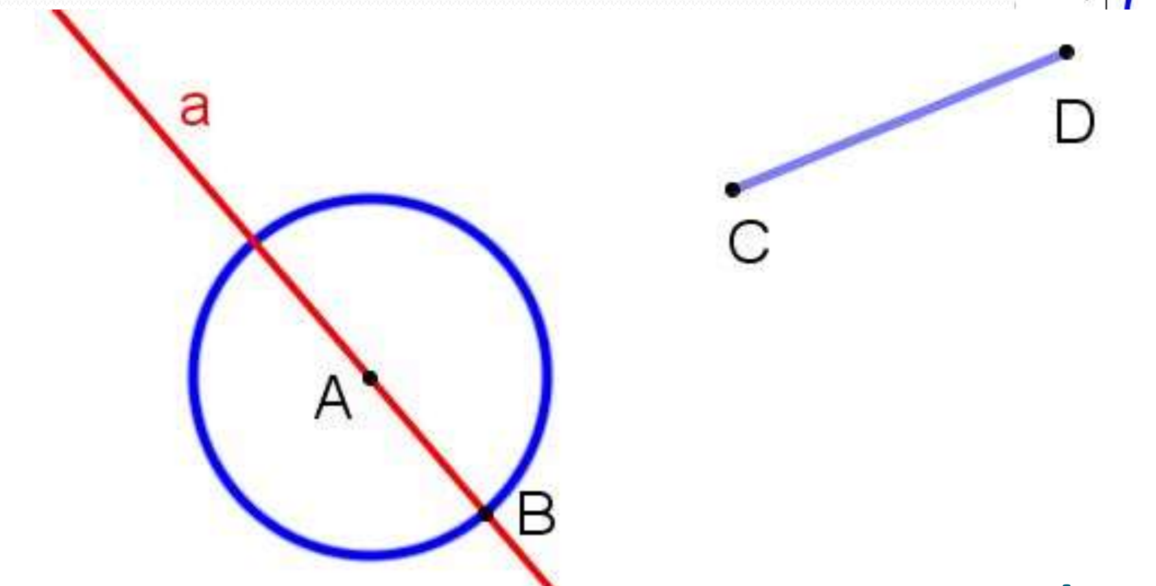
- Area
- Multiplication
- Proportions

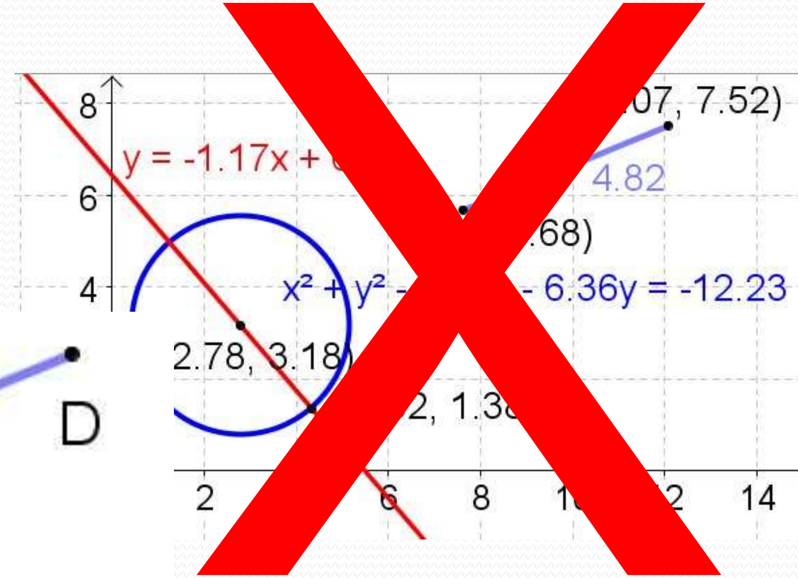


$$\frac{a}{b} = \frac{c}{d}$$

- And for that we develop our arithmetic:

But We Only Have Shapes…





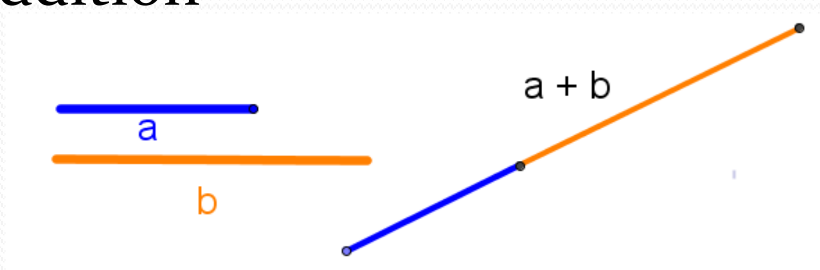
(…not numbers!)

=> Arithmetic on Segments

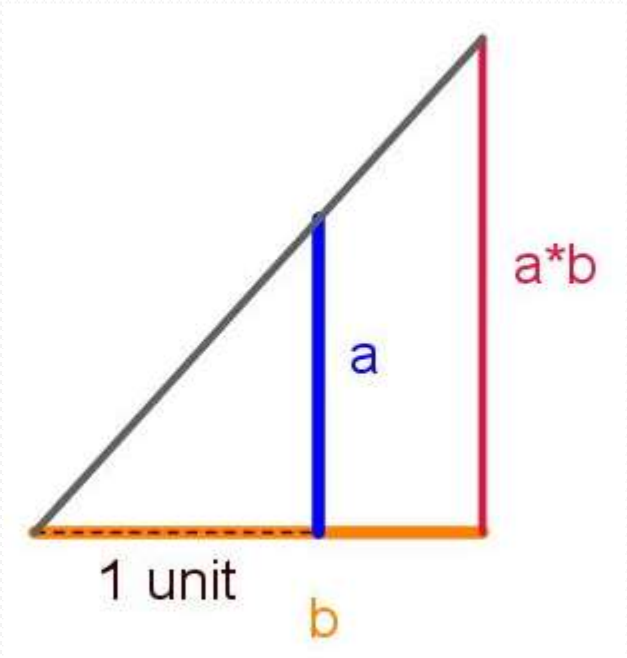
- Our numbers are

Equivalence Classes of Segments Under Congruence

- Addition

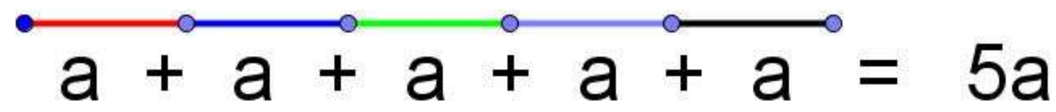


- Multiplication
  - We are <u>not</u> multiplying <u>numbers</u>
  - We are multiplying <u>segments</u>



Concept of Multiplication

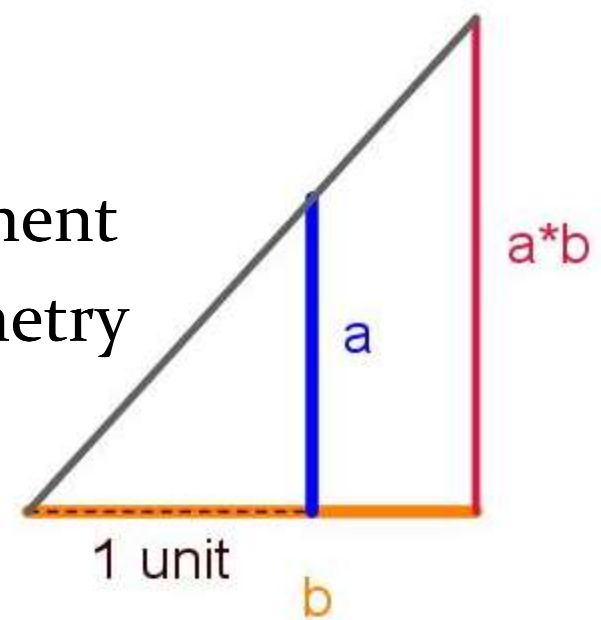
- Although there is repeated addition…



…this multiplication is different:

- The <u>product of two segments</u> is a segment
- Intuitively from similarity or trigonometry
- Difficult for teachers:

  we're defining, not deducing

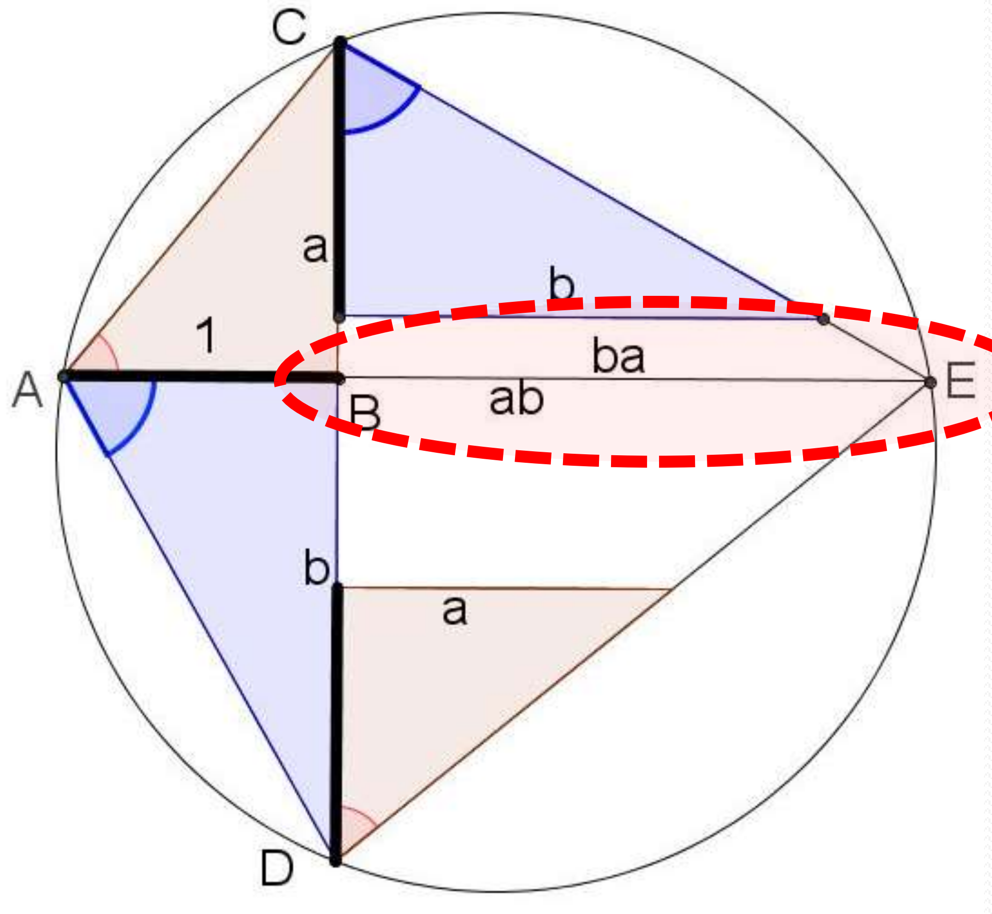


"…a different way to multiply […] really gives mastery…"

Proving the Field Properties

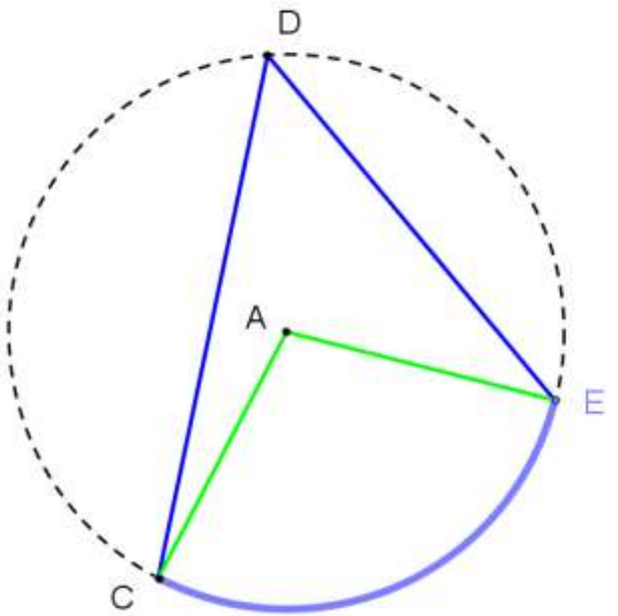
- Commutativity more difficult
- Requires interesting Geometry:
  - Inscribed/Central Angle Theorem:…

Hartshorne



BE = (a)(b) = (b)(a)

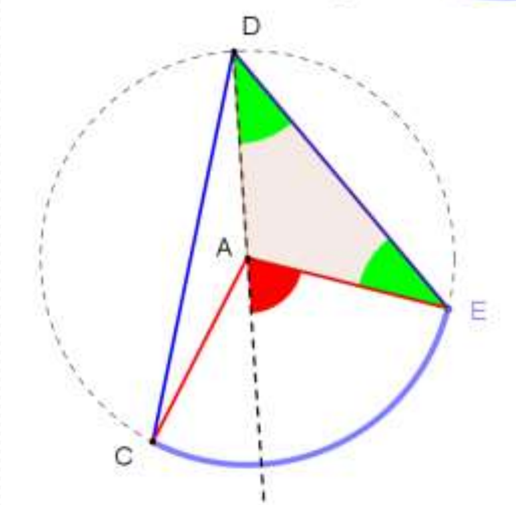
Inscribed and Central angles

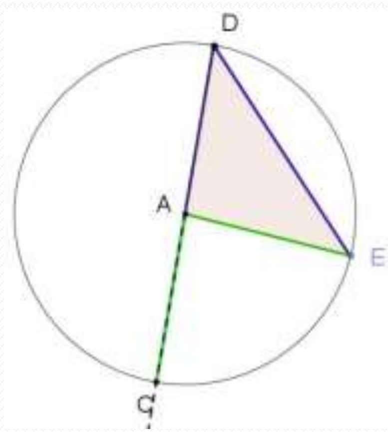


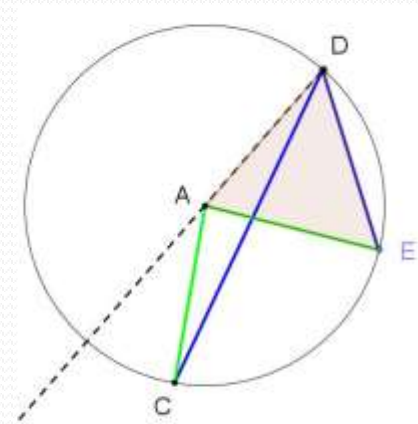
- One Proof Does it All…
- …or does it?

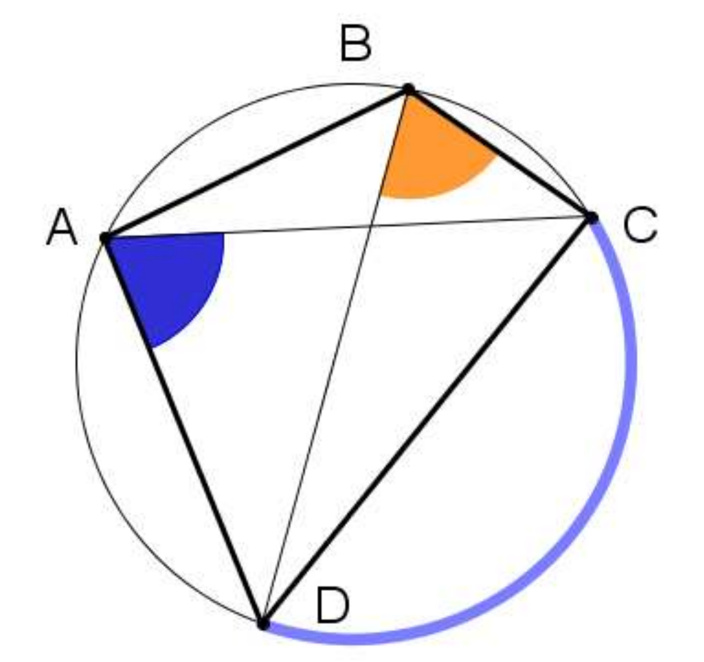
(InscrCentr.ggb)

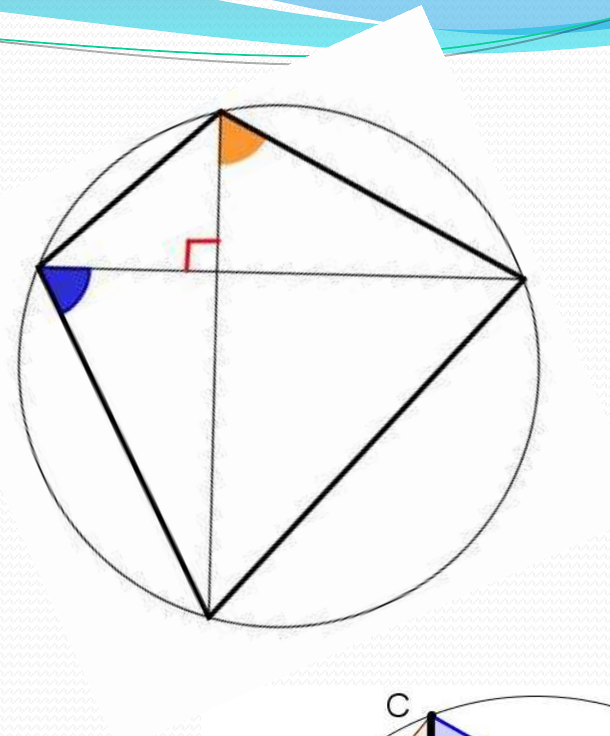
- Making the Case for Cases



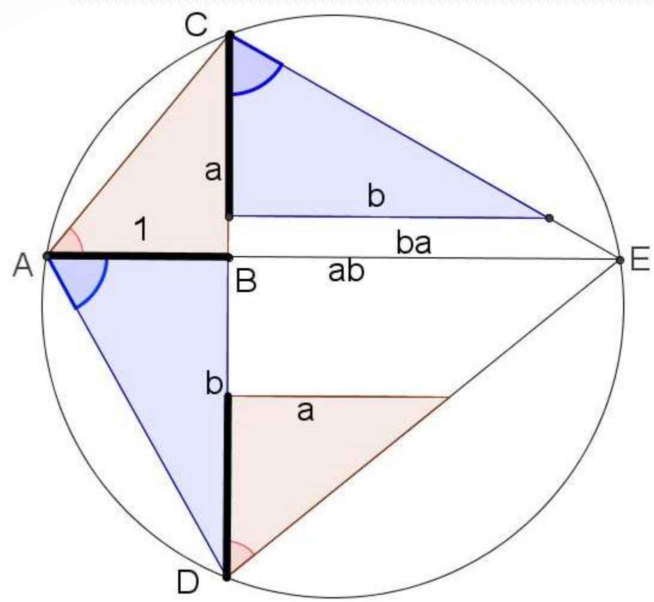






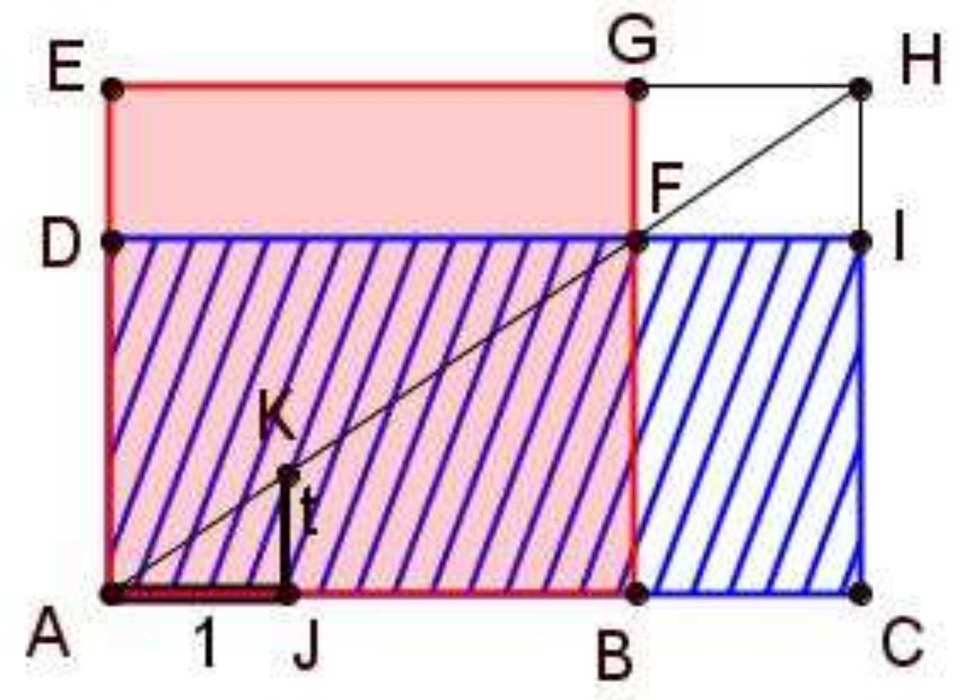


Cyclic Quadrilaterals



Rectangles of Equal Area

- Connection to segment multiplication:

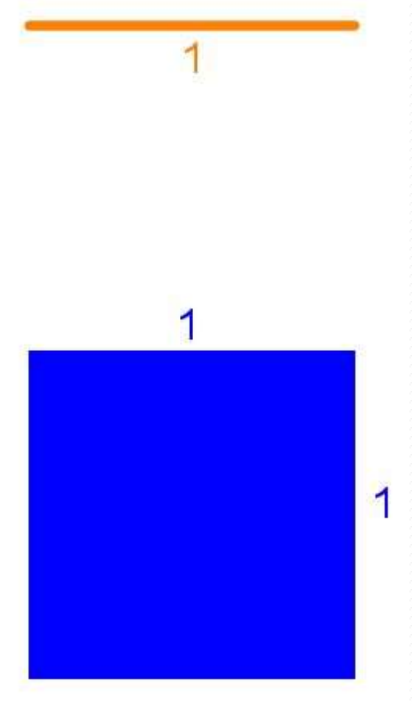


$$\left.\begin{array}{l} t(AB) = BF \\ CH = t(AC) \end{array}\right\} \rightarrow \text{t(AB)(CH)=t(BF)(AC)}$$

i.e. $t(L_1)(W_1) = t(L_2)(W_2)$

Choosing an Area Formula

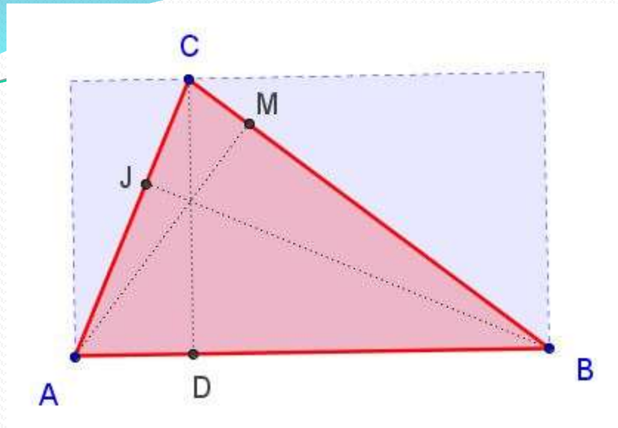
- **Area = t × Length × Width** seems a great choice
- But how do we choose "t"?
  - Like with length, we need a unit:
  - Choice: a square with unit sides
  - With this choice, **t = 1**

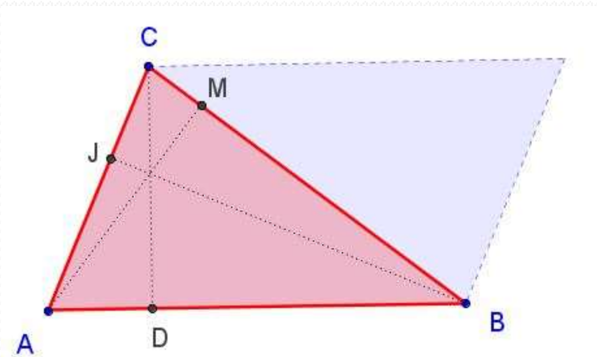


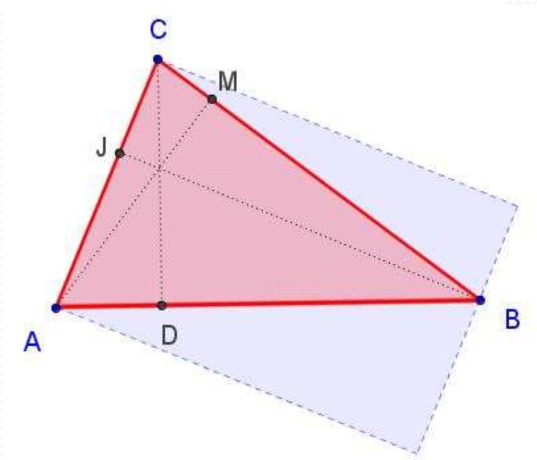
- $\mathbf{A = L \times W}$

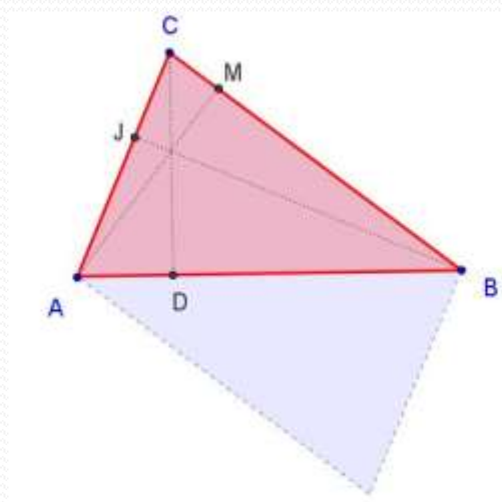
Areas of triangles

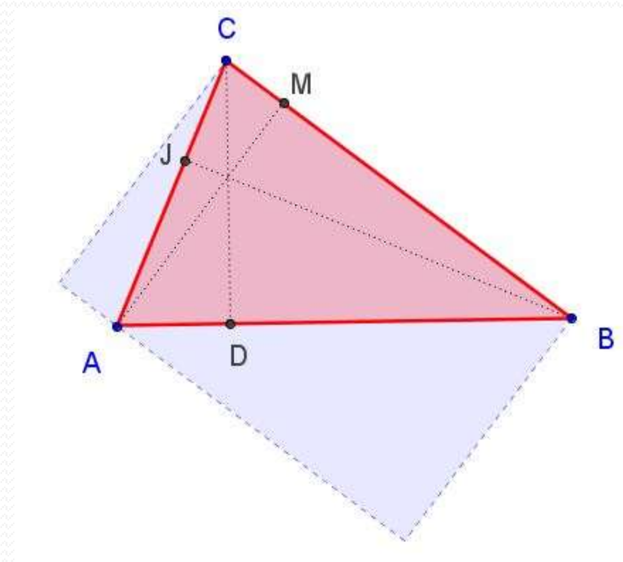
- Show that the formula $A = \frac{1}{2}bh$ makes sense,
- i.e. verify that choice of base does not matter

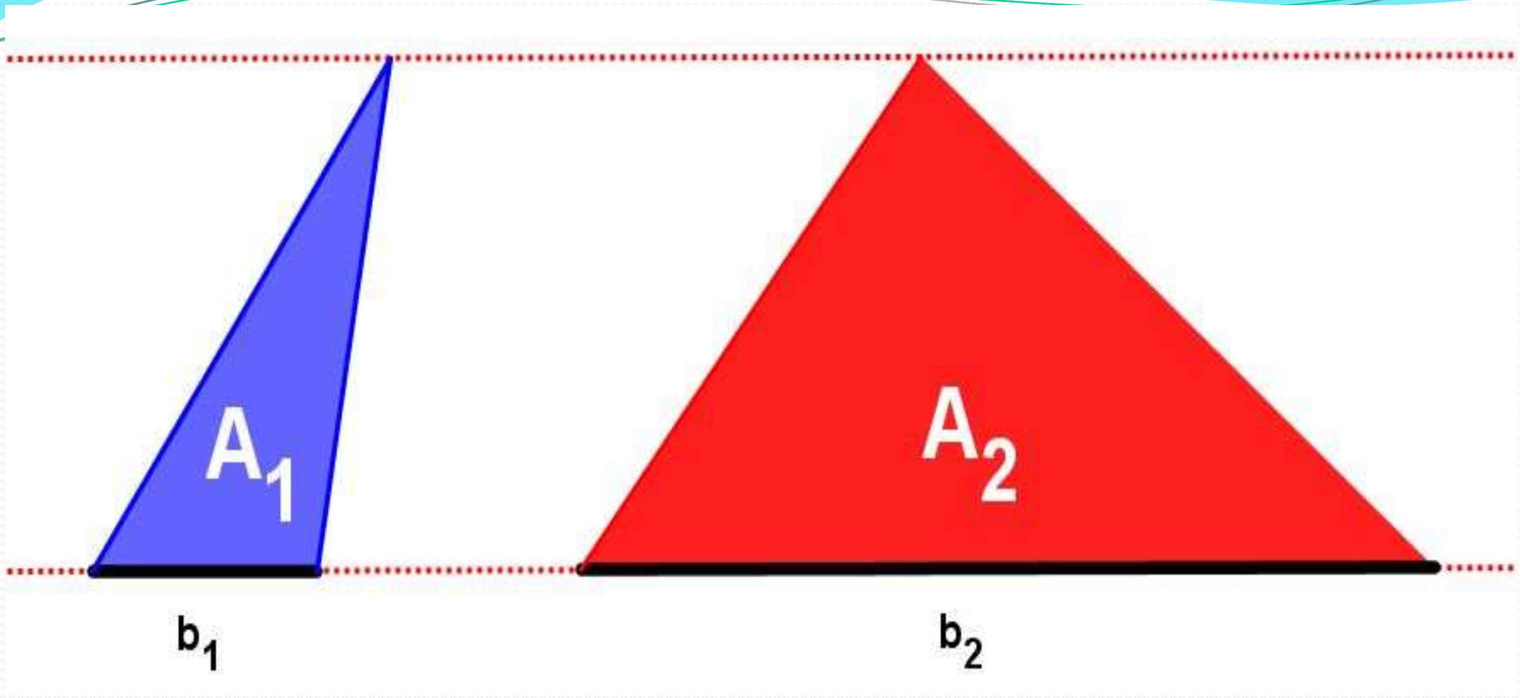










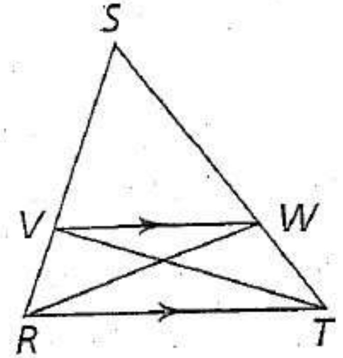


- Areas of triangles with equal heights are proportional to their bases

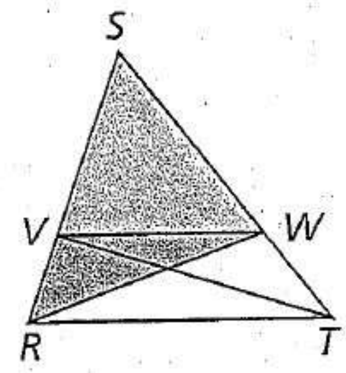
$$\frac{A_1}{A_2} = \frac{b_1}{b_2}$$

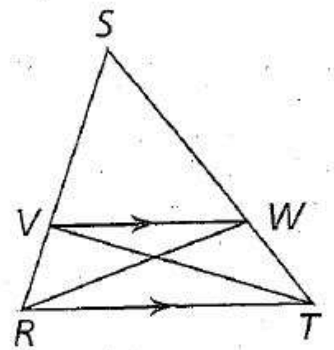
From

CME:

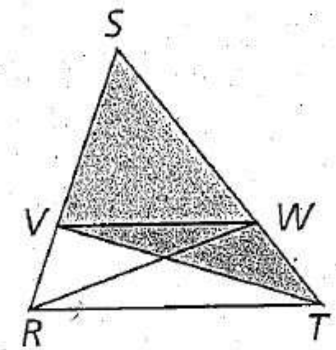


$$\frac{\text{area}(\triangle SVW)}{\text{area}(\triangle RVW)} = \frac{SV}{VR}$$





$$\frac{\text{area}(\triangle SVW)}{\text{area}(\triangle TVW)} = \frac{SW}{WT}$$



$$\frac{SV}{VR} = \frac{\text{area}(\triangle SVW)}{\text{area}(\triangle RVW)} = \frac{\text{area}(\triangle SVW)}{\text{area}(\triangle TVW)} = \frac{SW}{WT}$$

On Irrationals

- Now we have some irrational lengths:
- But we may not have others, like Pi, as in the

  Geometry over the Real Algebraic Numbers

- Arc Length is different from Segment Length
  (separate Undefined Terms in CCSSM)

- Ruler and Protractor Postulates
  (correspondence of lengths and angles with Reals)

Teacher Reactions

- "I learned [...]from other teachers"
- "increased my content knowledge"
- "providing multiple perspectives to a problem"
- "...embrace proof as a friend"
- "make math concepts interesting"
- "a different way to multiply [...] really gives mastery"
- "provided ideas"
- "I learned how to add/multiply using segments [...] this is something I've never considered doing, but is extremely useful."

THANK YOU

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