The Vaught Conjecture
Do uncountable models count?

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Is the Vaught Conjecture model theory ?

Possible simple answer: Yes, because it is true at certain levels in the first order stability hierarchy and fails at others.

Our approach. What specific model theoretic as opposed to descriptive set theoretic techniques can attack the problem?

Occam's razor

In particular we will investigate the differences between $L_{\omega,\omega}$ and $L_{\omega_1,\omega}$.

Vaught's conjecture concerns the set of countable models of a 'theory'.

An AEC is one of the most abstract formulations of 'theory'.

ABSTRACT ELEMENTARY CLASSES

Definition 1 *A class of L-structures,* $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$, *is said to be an* abstract elementary class: AEC *if both* $\boldsymbol{K}$ *and the binary relation* $\preceq_{\boldsymbol{K}}$ *are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If* $M \preceq_{\boldsymbol{K}} N$ *then* $M \subseteq N$.
- **A2**. $\preceq_{\boldsymbol{K}}$ *is a partial order on* $\boldsymbol{K}$.
- **A3**. *If* $\langle A_i : i < \delta \rangle$ *is* $\preceq_{\boldsymbol{K}}$*-increasing chain:*
  1. $\bigcup_{i<\delta} A_i \in \boldsymbol{K}$;
  2. *for each* $j < \delta$, $A_j \preceq_{\boldsymbol{K}} \bigcup_{i<\delta} A_i$
  3. *if each* $A_i \preceq_{\boldsymbol{K}} M \in \boldsymbol{K}$ *then* $\bigcup_{i<\delta} A_i \preceq_{\boldsymbol{K}} M$.

- **A4**. *If* $A, B, C \in \boldsymbol{K}$, $A \preceq_{\boldsymbol{K}} C$, $B \preceq_{\boldsymbol{K}} C$ *and* $A \subseteq B$ *then* $A \preceq_{\boldsymbol{K}} B$.
- **A5**. *There is a Löwenheim-Skolem number* $\mathrm{LS}(\boldsymbol{K})$ *such that if* $A \subseteq B \in K$ *there is a* $A' \in \boldsymbol{K}$ *with* $A \subseteq A' \preceq_{\boldsymbol{K}} B$ *and* $|A'| < \mathrm{LS}(\boldsymbol{K}) + |\mathrm{A}|$.

VC for AEC?

Does VC hold for AEC?

NO! The set $\boldsymbol{K} = \{\alpha : \alpha \leq \aleph_1\}$ with $\preceq_{\boldsymbol{K}}$ as initial segment is an AEC with $\aleph_1$ countable models.

But there are no large models. The upward Lowenheim-Skolem theorem is true for $L_{\omega,\omega}$ but not $L_{\omega_1,\omega}$.

Must a counterexample to VC in $L_{\omega_1,\omega}$ have a model of power $\aleph_2$ or even $\aleph_1$?

Can it have a model in $\aleph_2$?

Constructing models of larger power

$M \in \boldsymbol{K}$ is *maximal* if there is no $N \neq M$ with $M \preceq_{\boldsymbol{K}} N$.
Obviously,

Lemma 2 *In any AEC, if there is no maximal model of size λ, there is a model of size λ^+.*

And even

Lemma 3 *In any AEC, if there a strictly increasing $\preceq_{\boldsymbol{K}}$-sequence M_α, $\alpha < \lambda^+$ of models of size λ, there is a model of size λ^+.*

Some $L_{\omega_1,\omega}$ background

A model is *small* if it realizes only countably many $L_{\omega_1,\omega}$-types over the empty set.

M is small if and only M is Karp-equivalent to a countable model.

ϕ is complete for $L_{\omega_1,\omega}$ if for every sentence ψ of $L_{\omega_1,\omega}$, either $\phi \rightarrow \psi$ or $\phi \rightarrow \neg\psi$.

Note that a sentence is complete if and only if it is a Scott sentence; so every model of a complete sentence is small.

Definition 4 *A fragment* Δ *of* $L_{\omega_1,\omega}$ *is a subset of* $L_{\omega_1,\omega}$ *closed under subformula, substitutions of terms, finitary logical operations and such that: whenever*
$\Theta \subset \Delta$ *is countable and* $\phi, \bigvee \Theta \in \Delta$ *then*
$\bigvee\{\exists x\theta : \theta \in \Theta\}$,
$\bigvee\{\phi \wedge \theta : \theta \in \Theta\}$,
and $\bigvee(\{\phi\} \cup \Theta)$ *are all in* Δ.

Scattered Sentences

Definition 5 *A sentence σ of $L_{\omega_1,\omega}$ is* scattered *if for every countable fragment Δ of $L_{\omega_1,\omega}$, $S_n(\sigma, \Delta)$ is countable for each n.*

Theorem 6 (Morley) *If σ is a counterexample to VC, σ is scattered.*

Note this proof has descriptive set theoretic content.

If σ is scattered and $\sigma' \rightarrow \sigma$, then σ' is scattered.

Minimal sentences

Now we sketch the analysis of Harnik and Makkai [3] to show every counterexample to VC has an uncountable 'large' model.

Definition 7 *A sentence* σ *of* $L_{\omega_1,\omega}$ *is* large *if it has uncountably many countable models. A large sentence* σ *is* minimal *if for every sentence* ϕ *either* $\sigma \wedge \phi$ *or* $\sigma \wedge \neg\phi$ *is not large.*

Lemma 8 (Harnik-Makkai) *For every counterexample* σ *to Vaught's conjecture, there is a minimal counterexample* ϕ *such that* $\phi \models \sigma$*.*

Towards a Large uncountable model

Fix a minimal counterexample σ to Vaught's conjecture. For any countable fragment Δ containing σ, define

$$T_\Delta = \{\sigma \wedge \phi : \phi \in \Delta \text{ and } \sigma \wedge \phi \text{ is large } \}.$$

Note that T_Δ is consistent and complete for Δ.

Keisler [4] shows that the ‘prime’ part of Vaught’s fundamental paper goes through for scattered σ. In particular,

Fact 9 *A theory T that is complete for a countable fragment of $L_{\omega_1,\omega}$ and has only countably many pure types has a prime model.*

Since σ is scattered, each T_Δ has a prime model (for Δ).

The first construction

Lemma 10 *If σ is a counterexample to the Vaught Conjecture and Δ is smallest fragment containing σ, there is a strictly increasing $\prec_\Delta$-sequence M_α, $\alpha < \aleph_1$ of countable models.*

Proof. Fix a minimal counterexample σ to Vaught's conjecture and let Δ_0 be a countable fragment containing σ ($\{\sigma\} = T_0$). Define by induction $\langle \Delta_\alpha, T_\alpha, M_\alpha \rangle$ such that

1. If $\beta < \alpha$, the Scott sentence ψ_β of M_β is in Δ_α.
2. $T_\alpha = T_{\Delta_\alpha}$
3. M_α is the Δ_α prime model of T_α.

For this, let Δ_α be the minimal fragment containing $\bigcup_{\beta<\alpha} \Delta_\beta$ and the Scott sentence of each M_β for $\beta < \alpha$. The M_α are as required. The chain is strictly increasing since $M_\alpha \models \neg\psi_\beta$ if $\beta < \alpha$. And each $M_\alpha \prec_{\Delta_0} M_\beta$ for $\alpha < \beta$ since the Δ_i and T_i are increasing. That is, M_α is the prime model of T_α and $M_\beta \models T_\alpha$. $\square_{10}$

A Large uncountable model

Theorem 11 (Harnik-Makkai) *If $\sigma \in L_{\omega_1,\omega}$ is a counterexample to VC then it has a model N of cardinality $\aleph_1$ which is not small.*

Proof. We continue the argument from Lemma 10. Now if $M = \bigcup_\alpha M_\alpha$, M does not satisfy any complete sentence of $L_{\omega_1,\omega}$, as any sentence θ true on M is true on a cub of M_α so has more than one countable model. $\square_{11}$

Aside: Getting small uncountable models

Lemma 12 *A sentence σ of $L_{\omega_1,\omega}$ has an uncountable small model iff it has a pair of countable models such that M_0 is a proper substructure of M_1, M_0 and M_1 are isomorphic and $M_0 \prec_L M_1$, where L is the smallest fragment containing the Scott sentence of M_0.*

Proof. If N is an uncountable small model of σ, let ψ be the Scott sentence of N and L the fragment generated by ψ. Then take M_0 an L-elementary submodel of N and M_1 an L-elementary submodel of N which properly extends M_0. Conversely, construct an chain $M_i : i < \aleph_1$ where (M_i, M_{i+1}) is isomorphic to (M_0, M_1). This construction goes through limits by taking unions since for countable δ, all M_δ are isomorphic. $\square_{12}$

Is there any direct way (using only countable models) to deduce the countable model version of this directly from a failure of VC?

REPLY: YES (Sacks) using admissible sets and Barwise compactness and a nice argument.

And in essence Makkai's original argument.

Bounds on well-orders

We rely on the following result which combines results of Lopez-Escobar, Morley, and Keisler. The ingredients are in [4].

Theorem 13 *Let τ be a similarity type which includes a binary relation symbol $<$. Suppose ψ is a sentence of $L_{\omega_1,\omega}(Q)$, $M \models \psi$, and the order type of $(M,<)$ imbeds ω_1. There is a model N of ψ with cardinality $\aleph_1$ such that the order type of $(N,<)$ imbeds $\mathbb{Q}$.*

Constructing Small Uncountable Models

The proof of the next lemma is due to Shelah [6] (see Section 7.3 of [1]). Note that the hypothesis is satisfied by any scattered $L_{\omega_1,\omega}$-sentence that has an uncountable model.

Theorem 14 *If the $L_{\omega_1,\omega}$-τ-sentence ψ has a model of cardinality $\aleph_1$ which is L^*-small for every countable τ-fragment L^* of $L_{\omega_1,\omega}$, then ψ has a τ-small model of cardinality $\aleph_1$.*

Add to τ a binary relation $<$, interpreted as a linear order of M with order type ω_1. Using that M realizes only countably many types in any τ-fragment, write $L_{\omega_1,\omega}(\tau)$ as a continuous increasing chain of fragments L_α such that each type in L_α realized in M is a formula in $L_{\alpha+1}$.

Extend the similarity type to τ' by adding new $2n+1$-ary predicates $E_n(x, \overline{y}, \overline{z})$ and $n+1$-ary functions f_n.

Let M satisfy $E_n(\alpha, \overline{a}, \overline{b})$ if and only if $\overline{a}$ and $\overline{b}$ realize the same Δ_α-type.

Let f_n map M^{n+1} into the initial ω elements of the order, so that $E_n(\alpha, \overline{a}, \overline{b})$ implies $f_n(\alpha, \overline{a}) = f_n(\alpha, \overline{b})$.

Some Facts

1. $E_n(\beta, \overline{y}, \overline{z})$ refines $E_n(\alpha, \overline{y}, \overline{z})$ if $\beta > \alpha$;
2. $E_n(0, \overline{a}, \overline{b})$ implies $\overline{a}$ and $\overline{b}$ satisfy the same quantifier free τ-formulas;
3. If $\beta > \alpha$ and $E_n(\beta, \overline{a}, \overline{b})$, then for every c_1 there exists c_2 such that $E_{n+1}(\alpha, c_1\overline{a}, c_2\overline{b})$ and
4. f_n witnesses that for any $a \in M$ each equivalence relation $E_n(a, \overline{y}, \overline{z})$ has only countably many classes.

All these assertions can be expressed by an $L_{\omega_1,\omega}(\tau`)$ sentence $\phi$. Let $\Delta^*$ be the smallest $\tau`$-fragment containing $\phi \wedge \psi$.

Now by Lopez-Escobar (Theorem 13) there is a structure N of cardinality $\aleph_1$ satisfying $\phi \wedge \psi \wedge \chi$ such that $<$ is not well-founded on N.

Fix an infinite decreasing sequence $d_0 > d_1 > \ldots$ in N. For each n, define $E_n^+(\overline{x}, \overline{y})$ if for some i, $E_n(d_i, \overline{x}, \overline{y})$. Now using 1), 2) and 3) prove by induction on the quantifier rank of ϕ for every $L_{\omega_1,\omega}(\tau)$-formula ϕ that

$N \models E_n^+(\overline{a}, \overline{b})$ implies

$N \models \phi(\overline{a})$ if and only if $N \models \phi(\overline{b})$.

For each n, $E_n(d_0, \overline{x}, \overline{y})$ refines $E_n^+(\overline{x}, \overline{y})$ and by 4) $E_n(d_0, \overline{x}, \overline{y})$ has only countably many classes; so N is small. $\Box_{14}$

A small uncountable model

We conclude the result proved by Makkai[5] and by Sacks using admissible model theory.

Theorem 15 (Makkai) *If $\sigma \in L_{\omega_1,\omega}$ is a counterexample to VC then it has an uncountable model N which is small.*

Proof. By Lemma 6, ψ is scattered. By Theorem 11, it has a model of power $\aleph_1$ and then by Lemma 14, it has a small uncountable model. $\square_{15}$

Corollary 16 *There is no $\aleph_1$-categorical counterexample to Vaught's conjecture.*

The number of models in $\aleph_1$

Theorem 17 *If a first order theory is a counterexample to the Vaught conjecture then it has $2^{\aleph_1}$ models of cardinality $\aleph_1$.*

This is easy from two difficult theorems:

Theorem 18 (Shelah) *If a first order T is not ω-stable T has $2^{\aleph_1}$ models of cardinality $\aleph_1$.*

This argument uses many descriptive set theoretic techniques. See Shelah's book [7] or Baldwin's paper [2].

Theorem 19 (Shelah) *An ω-stable first order theory satisfies Vaught's conjecture.*

The number of models in $\aleph_1$: $L_{\omega_1,\omega}$

Question 20 *Does the previous theorem extend to $L_{\omega_1,\omega}$?*

Keisler showed

Theorem 21 *For any $L_{\omega_1,\omega}(Q)$-sentence ψ and any fragment L^* containing ψ, if ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then for any $M \models \psi$ of cardinality $\aleph_1$, M realizes only countably many L^*-types over the empty set*

Shelah observed that Theorem 21 implies:

Fact 22 *($2^{\aleph_0} < 2^{\aleph_1}$) If a sentence $\psi \in L_{\omega_1,\omega}$ is not ω-stable it has $2^{\aleph_1}$ models of cardinality $\aleph_1$.*

For first order logic:
Few models in $\aleph_1$ implies ω-stable.

But for $L_{\omega_1,\omega}$.
Few models in $\aleph_1$ implies ω-stable, requires weak CH.

This leads us to: Does VC hold for ω-stable sentences in $L_{\omega_1,\omega}$? For excellent classes?

AMALGAMATION

Theorem 23 (Shelah) *If a sentence σ in $L_{\omega_1,\omega}$ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then the countable models of σ have the amalgamation property.*

Chapter 8 of my monograph.

‘Complete’ sentences

Note complete vrs Δ-complete (where Δ is a countable fragment).

A complete sentence of $L_{\omega_1,\omega}$ is $\aleph_0$-categorical, trivializing Vaught’s conjecture.

But Shelah ‘reduces’ Morley’s theorem for $L_{\omega_1,\omega}$ to complete sentences. This reduction involves a further crucial model theoretic technique:

Prove a theorem for arbitrary vocabularies τ.

Theorem 24 *Let* ψ *be a* complete *sentence in* $L_{\omega_1,\omega}$ *in a countable vocabulary* τ. *Then there is a countable vocabulary* τ' *extending* τ *and a first order* τ'*-theory* T *such that reduct is a* 1-1 *map from the* atomic *models of* T *onto the models of* ψ.

If ψ is not complete, the reduction is only to 'finite diagrams'.

THE REDUCTION: Arbitrarily large models

Theorem 25 *Let ψ be an $L_{\omega_1,\omega}(\tau)$-sentence which has arbitrarily large models. If ψ is categorical in some cardinal κ, ψ is implied by a consistent complete sentence ψ', which has a model of cardinality κ.*

This is a fairly straightforward argument with Ehrenfeucht-Mostowski models.

THE REDUCTION: $\aleph_1$

Theorem 26 *Let ψ be an $L_{\omega_1,\omega}(\tau)$-sentence If ψ is categorical in $\aleph_1$, ψ is implied by a consistent complete sentence ψ', which has a model of cardinality $\aleph_1$.*

Proof. If not, by Theorem 21, there is a model of power $\aleph_1$ which is L^*-small for every countable fragment L^*. But then by Theorem 14, the model is actually small.

Theorem 27 *[Shelah]*

1. *(For $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$) A complete $L_{\omega_1,\omega}$-sentence which has few models in $\aleph_n$ for each $n < \omega$ is excellent.*
2. *(ZFC) An excellent class has models in every cardinality.*
3. *(ZFC) Suppose that ϕ is an excellent $L_{\omega_1,\omega}$-sentence. If ϕ is categorical in one uncountable cardinal κ then it is categorical in all uncountable cardinals.*

1. Weak GCH and categoricity up $\aleph_\omega$ implies excellence.
2. Categoricity up to $\aleph_n$ does not suffice.

Corollary 28 *Suppose* $\boldsymbol{K}$ *is *-excellent. If* $\boldsymbol{K}$ *is not* $\aleph_1$ *categorical, then* $\boldsymbol{K}$ *has at least* $n+1$ *models of cardinality* $\aleph_n$ *for each* $n < \omega$.

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