Constructing Borel Models in the Continuum
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Today's Topics

Section 1: Context for this seminar

Models in $L_{\omega_1,\omega}$

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$L_{\omega_1,\omega}$ satisfies downward Löwenheim Skolem to $\aleph_0$ for sentences.
It does not satisfy upward Löwenheim Skolem.

Problem

Find sufficient conditions on countable models of a sentence ϕ to guarantee a model in

1. $\aleph_1$
2. $2^{\aleph_0}$
3. arbitrary cardinals

Sources

Shelah: 522 and 1098 and the bible

Baldwin-Laskowski - Building models in the Continuum
Ackerman, Freer, Patel
comments by Phil Wesolek

See related work by Scow, Kim, Kim, Tsuboi, Laskowski-Shelah on 'generalized indiscernability'.

From $L_{\omega_1,\omega}$ to 'first order'

From $L_{\omega_1,\omega}$ to 'first order'

1. $\phi \in L_{\omega_1,\omega} \to (T, \Gamma)$
2. complete $\phi \in L_{\omega_1,\omega} \to (T, \mathit{Atomic})$

The translation

Theorem

[Chang/Lopez-Escobar] Let ψ be a sentence in $L_{\omega_1,\omega}$ in a countable vocabulary τ. Then there is a countable vocabulary τ' extending τ, a first order τ'-theory T, and a countable collection of τ'-types Γ such that reduct is a 1-1 map from the models of T which omit Γ onto the models of ψ.

The proof is straightforward. E.g., for any formula ψ of the form $\bigwedge_{i<\omega} \phi_i$, add to the language a new predicate symbol $R_\psi(\mathbf{x})$. Add to T the axioms

$$(\forall \mathbf{x})[R_\psi(\mathbf{x}) \rightarrow \phi_i(\mathbf{x})]$$

for $i < \omega$ and omit the type $p = \{\neg R_\psi(\mathbf{x})\} \cup \{\phi_i : i < \omega\}$.

Δ-complete

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.
ϕ is Δ-complete if for every $\psi \in \Delta$
$\phi \models \psi$ or $\phi \models \neg\psi$.
(If Δ is omitted we mean complete for $L_{\omega_1,\omega}$.)

small-complete

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.

Definition

A τ-structure M is Δ-small if M realizes only countably many Δ-types (over the empty set).

'small' means $\Delta = L_{\omega_1,\omega}$

Reducing complete to atomic

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
K is the class of atomic models (realize only principal types) of a first order theory (in an expanded language).

GEOMETRIES

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If points are closed the structure is called a geometry.

STRONGLY MINIMAL I

M is strongly minimal if every first order definable subset of any elementary extension M' of M is finite or cofinite.

$a \in \mathrm{acl}(B)$ if for some $\mathbf{b} \in B$ and some $\phi(x, \mathbf{y})$: $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only **finitely many** solutions.

STRONGLY MINIMAL II

A complete first order theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

Section 2: Asymptotic Similarity

Asymptotic Similarity

Notation: Indexing by Trees

1. For any $\overline{\eta} = \eta_1, \ldots \eta_k$ and $\overline{\eta}' = \eta'_1, \ldots \eta'_k$ sequences from $2^{\leq\omega}$, $\overline{\eta}$ and $\overline{\eta}'$ are <u>similar</u> over n if
   a) Each $\lg(\eta_i), \lg(\eta'_i) \geq n$;
   b) for each $i \leq k, \eta_i \restriction n = \eta'_i \restriction n$:
   c) for each $i < j \leq k, \eta_i \restriction n \neq \eta_j \restriction n$.
2. For $\overline{\eta} = \eta_1, \ldots \eta_k$ a finite sequence from $2^{\leq\omega}$ and $\{\boldsymbol{a}_\eta : \eta \in 2^{<\omega}\}$ a set of m-tuples indexed by $2^{\leq\omega}$, $\boldsymbol{a}_{\overline{\eta}}$ denotes $\boldsymbol{a}_{\eta_1}, \ldots \boldsymbol{a}_{\eta_k}$.

m will be finite or ω or changing for various applications.

Asymptotic Similarity

Definition

The set of m-tuples, $\{\boldsymbol{a}_\eta : \eta \in 2^\omega\}$ are asymptotically similar if for every formula $\phi(\mathbf{y}_1, \ldots \mathbf{y}_k)$ where each $\mathbf{y}$ has length m, there is an n_ϕ such that
for any $\overline{\eta} = \eta_1, \ldots \eta_k$ and $\overline{\eta}' = \eta'_1, \ldots \eta'_k$ and any $n \geq n_\phi$ if $\overline{\eta}$ and $\overline{\eta}'$ are similar over n,

$$\phi(\boldsymbol{a}_{\overline{\eta}}) \leftrightarrow \phi(\boldsymbol{a}_{\overline{\eta}'}).$$

Not indiscernible. From Shelah VII.3.7
Application: Non-ω-stable theories have at least $\min(2^\mu, \beth_2)$ models in μ.

Two extensions

1. The asymptotically similar set $\{\boldsymbol{a}_\eta : \eta \in 2^\omega\}$ will be approximated by $\{\boldsymbol{a}_{\eta\restriction k} \in 2^{<\omega}\}$;

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1. The asymptotically similar set $\{\boldsymbol{a}_\eta : \eta \in 2^\omega\}$ will be approximated by $\{\boldsymbol{a}_{\eta\restriction k} \in 2^{<\omega}\}$;
2. the m-tuples $\boldsymbol{a}_{\eta\restriction n}$ will be replaced by m_n-sequences whose length depends on n.

More Notation

Notation

$\boldsymbol{a}_{\eta\restriction n} = \langle a_\eta^0, \ldots a_\eta^{m_n} \rangle$ where the $m_n = m_n(\eta)$ are a non-decreasing sequence.

We will be describing below the diagram of a model with universe

$$A = \{a_\eta^i : i < \omega, \eta \in 2^\omega\}.$$

We will do that with a stock of continuum many variables y_η^i where $i \in \omega$ and $\eta \in 2^{\leq\omega}$.

Finite Asymptotic Similarity

$\phi(\mathbf{y}_1, \ldots \mathbf{y}_k)$ is $\phi(y^0_{\eta_0}, \ldots y^{m_\phi}_{\eta_0}, y^0_{\eta_1}, \ldots y^{m_\phi}_{\eta_1}, \ldots y^0_{\eta_k}, \ldots y^{m_\phi}_{\eta_k})$

Definition

The tuples in $A = \{\boldsymbol{a}_\eta : \eta \in 2^{<\omega}\}$, are finitely asymptotically similar if for every formula $\phi(\mathbf{y}_1, \ldots \mathbf{y}_k)$, there is an n_ϕ such that for any $\overline{\eta} = \eta_1, \ldots \eta_k$ and $\overline{\eta}' = \eta'_1, \ldots \eta'_k$, each with domain at least n_ϕ, and any $n \geq n_\phi$ if $\overline{\eta}$ and $\overline{\eta}'$ are similar over n,

$$\phi(\boldsymbol{a}_{\overline{\eta}}) \leftrightarrow \phi(\boldsymbol{a}_{\overline{\eta}'}).$$

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Gloss on Shelah: VII.3.7

Theorem VII.3.7

If a countable complete theory has uncountably many types over the empty set then it has a finitely asymptotic sequence.

By the Morley splitting argument and Halpern-Lauchli.

Application: Shelah

If a first order T is not small there $\min(2^{\mu}, \beth_1)$ models in μ.

See also my paper Diverse Classes

Finite approximations to the continuum

Lemma

If $\{\boldsymbol{a}_\eta : \eta \in 2^{<\omega}\}$ is a finitely asymptotically similar tree on $2^{<\omega}$, then there is an asymptotically similar tree on 2^ω.

Proof. For $\eta \in 2^\omega$, define $\phi(\boldsymbol{a}_{\overline{\eta}})$ to hold if and only for some $n \geq n_\phi$ with the $\eta_i{\restriction}n$ distinct, $\phi(\boldsymbol{a}_{\overline{\eta}}{\restriction}n)$ holds. By the asymptotic similarity $\phi(\boldsymbol{a}_{\overline{\eta}}{\restriction}r)$ holds for all $r \geq n$. By compactness, we have defined a consistent diagram of the $\{\boldsymbol{a}_\eta : \eta \in 2^\omega\}$.

The Limit Structure I

The limit structure A is somewhat peculiar. The entire structure is an inverse limit but each leaf $(\mathbf{a}_\eta)$ is direct limit.

Inverse Limit

Let $A = \{\mathbf{a}_\eta : \eta \in 2^\omega\} = \{a^i_\eta : i \in \omega, \eta \in 2^\omega\}$.
$A_n = \{\mathbf{a}_\eta : \eta \in 2^n\}$.
To consider A as an inverse limit, consider the maps f_{mn} (for $m \geq n$) which map A_m onto A_n by restricting $\tau \in 2^m$ to $\tau \restriction n$. These are homomorphisms and A is the inverse limit of the A_n under this system of surjections.

The Limit Structure II

Not quite standard; we partition the sequence at each node.

Borel topology

For $s \in 2^{<\omega}$, let $A_{s,i} = \{a^i_\eta : s \sqsubseteq \eta\}$. Now define the (standard) Borel topology on A by setting each $A_{s,i}$ to be open. We use the product topology on A^n so $s_0 \ldots s_k \in 2^{<\omega}$ and $\mathbf{i} = \langle i_0, \ldots i_k \rangle$, $A_{\mathbf{s},\mathbf{i}} = \{a^{i_0}_{\eta_0}, \ldots a^{i_k}_{\eta_k} : s_i \sqsubset \eta_i\}$ is open.

Note also that if we fix an $\eta \in 2^\omega$, $\boldsymbol{a}_\eta$ is the direct limit of the $\boldsymbol{a}_{\eta \restriction n}$ as n goes to ω.

The limit structure is Borel

Fact

In a relational vocabulary τ each $A_n = \{\boldsymbol{a}_\eta : \eta \in 2^n\}$ is a partial τ-structure (not all relations may be defined) but the asymptotically similar structure is a Borel τ-structure.

Proof. For $\eta \in 2^\omega$, define $\phi(\boldsymbol{a}_{\overline{\eta}})$ to hold if and only for some $n \geq n_\phi$ with the $\eta_i \restriction n$ distinct, $\phi(\boldsymbol{a}_{\overline{\eta}} \restriction n)$ holds. By the asymptotic similarity $\phi(\boldsymbol{a}_{\overline{\eta}} \restriction r)$ holds for all $r \geq n_\phi$. By compactness, we have defined a consistent diagram (i.e a finitely satisfiable set of formulas) of the $\{\boldsymbol{a}_\eta : \eta \in 2^\omega\}$.

The limit structure is Borel: detail

Fix a predicate symbol $R(x_1, \ldots, x_n)$. Let $\mathbf{i} = \langle i_1, \ldots i_n \rangle$ be a sequence of natural numbers. Abusing notation, we write $\mathbf{s} \in 2^m$ for $\langle s_1, \ldots s_n \rangle \in (2^m)^n$ where each $s_i \in 2^m$. Then $\boldsymbol{a}^{\mathbf{i}}_{\mathbf{s}}$ denotes $\langle a^{i_1}_{s_1} \ldots a^{i_n}_{s_n} \rangle$. Similarly, $\boldsymbol{a}^{\mathbf{i}}_{\overline{\eta}}$ denotes $\langle a^{i_1}_{\eta_1} \ldots a^{i_n}_{\eta_n} \rangle$.
There is a minimal $k_{\mathbf{i}}$ such that $R(a^{i_1}_{\eta_1 \restriction k} \ldots a^{i_n}_{\eta_n \restriction k})$ is defined for all $\overline{\eta}$. Let $t_i = \eta_i \restriction k$.
Note that $A_{\mathbf{t},\mathbf{i}} =_{df} A^R_{\mathbf{i}} = \{\overline{\eta} : A_{\mathbf{k}_{\mathbf{i}}} \models R(\boldsymbol{a}^{\mathbf{i}}_{\overline{\eta} \restriction k_{\mathbf{i}}})\}$ which is open in the Borel topology on A.
Then the set of $\langle a^{i_1}_{\eta_1} \ldots a^{i_n}_{\eta_n} \rangle \in A^n$ such that $\models R(a^{i_1}_{\eta_1} \ldots a^{i_n}_{\eta_n})$ is $\bigcup_{\mathbf{i}} A^R_{\mathbf{i}}$ which is open in the Borel topology on A. Thus A is a Borel structure.

Section 3: The General Template

Tree chunks

Notation

Let Y be the collection of all sets v of finite sequences of 0 or 1 such that the members of v are pairwise incompatible and for some n, all members of v are of length n or $n+1$. We call the set $\{n, n+1\}$, $\mathrm{dom}(v)$.

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Indexing by tree chunks and formulas: context

We will build diagrams of sets of finite asymptotically similar tuple indexed by sequences $\mathbf{z} = \mathbf{rxy}$.

1. AFP - only $\mathbf{x}$
2. quasiminimal, pseudominimal – $\mathbf{xy}$
3. almost pseudominimal – $\mathbf{rxy}$

Indexing by tree chunks and formulas: details

Notation

Let Z be the set of pairs $(v, \phi(\ldots, \mathbf{z}_\eta, \ldots))$ where

a) ϕ is a complete τ-formula with free variables in $\{\mathbf{z}_\eta : \eta \in v\}$ with $\mathbf{z}_\eta = \mathbf{r}_\eta x_\eta \mathbf{y}_\eta$. For convenience we will write $\mathbf{x}_\eta$ for $\mathbf{r}_\eta x_\eta$.

b) $v \in Y$.

c) For every v, $\phi(\ldots, \mathbf{z}_\eta, \ldots)_{\eta \in v}$ implies the $\langle \mathbf{r}_\eta x_\eta : \eta \in v\rangle$ realize the type of an independent sequence.

d) For each i and ξ, $y^i_\xi \in \mathrm{cl}(\mathbf{r}_\eta x_\eta : \eta \in v)$.

Successors

Definition

$(v', \phi') \in Z$ is a successor of $(v, \phi) \in Z$ at η if

1. v' is the result of replacing one sequence $\eta \in v$ by $\eta\hat{}0$ and $\eta\hat{}1$ and otherwise $\xi' = \xi$
2. $\phi'(\ldots, \mathbf{z}_\xi, \mathbf{z}_{\eta\hat{}0}, \mathbf{z}_{\eta\hat{}1}, \ldots,)_{\xi \in v, \xi \neq \eta} \rightarrow \phi(\ldots, \mathbf{z}_\xi, \mathbf{z}_{\eta\hat{}0}, \ldots,)_{\xi \in v, \xi \neq \eta}$
3. $\phi'(\ldots, \mathbf{z}_\xi, \mathbf{z}_{\eta\hat{}0}, \mathbf{z}_{\eta\hat{}1}, \ldots,)_{\xi \in v, \xi \neq \eta} \rightarrow \phi(\ldots, \mathbf{z}_\xi, \mathbf{z}_{\eta\hat{}1}, \ldots,)_{\xi \in v, \xi \neq \eta}$
4. The transitive closure of ‘successor’ defines a partial order on Z denoted by $<^*$.

Partial orders

Definition

We consider partially ordered sets $(\mathbb{P}, \leq)$ where $\mathbb{P}$ is a set of first order formulas in $\tau(T)$ and $p \leq q$ means $q \vdash p$.

A poset $(\mathbb{P}, \leq)$ is Z-compatible if there is a function $\Phi : \mathbb{P} \mapsto Z$ such that for any $p, q \in \mathbb{P}$, $p \leq_{\mathbb{P}} q$ implies $\Phi(p) \leq_Z \Phi(q)$.
We denote the formula component of $\Phi(p)$ by $\Phi_1(p)$.

Suitable Partial orders

Definition

We will say a Z-compatible $\mathbb{P}$ is suitable for T if it satisfies the following conditions.

Extendability For all $(v, \phi(\mathbf{z})) \in Z$ and all consistent (with T) $\psi(\mathbf{z})$ such that $\psi(\mathbf{z}) \vdash \phi(\mathbf{z})$, for every $p \in \Phi^{-1}(v, \phi(\mathbf{z}))$ there is a $q \in \mathbb{P}$ with $q \geq p$ such that $\Phi_1(p) \vdash \psi(\mathbf{z})$.

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Henkin witnesses For all $(v, \phi(\mathbf{z})) \in Z$ and all consistent $(\exists u)\psi(\mathbf{z}, u)$ such that $\psi(\mathbf{z}, u) \vdash \phi(\mathbf{z})$, for every $p \in \Phi^{-1}(v, \phi(\mathbf{z}))$ there is a $q \in \mathbb{P}$ with $q \geq p$ such that $\Phi_1(q) \vdash \psi(\mathbf{z}, u)$.

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splitting For all $(v, \phi(\mathbf{z})) \in Z$ and all $v_i \in v$ for every $p \in \Phi^{-1}(v, \phi(\mathbf{z}))$ there is a $q \in \mathbb{P}$ with $q \geq p$ with $\Phi(q) = (v', \psi)$, v' is a successor of v at v_i and where $\psi \vdash \phi$.

Γ

We define below a collection Γ of formulas ϕ_n, the description (at stage n) of the sequence indexed by v.

(v_{n+1}, ϕ_{n+1}) is a successor of (v_n, ϕ_n). If $m > n$, ϕ_m may have vastly more variables than ϕ_n and the variable sets may even be disjoint.
Suppose $\eta_1, \ldots \eta_k \in v_m$ and η_i^* is the restriction of η_i which is in v_n. Let $\mathbf{z}^*_{\eta_i^*} = \langle x_{\eta_i}, z^0_{\eta_i} \ldots z^r_{\eta_i} \rangle$ where r is the length of $\mathbf{z}_{\eta_i \restriction s}$, where $s = \max(\mathrm{dom}\, v_n)$. So $z^j_{\eta_i}$ in ϕ_m corresponds to $z^j_{\eta_i^*}$ in ϕ_n. Crucially, $\phi_n(\ldots x_{\eta_i} z^j_{\eta_i^*} \ldots)_{\eta_i^* \in v_n, j < r_n} \in \mathrm{cn}(\Gamma)$. But we want to transfer this information about variable indexed by sequences in v_n to substitutions in ϕ_n by the $z^j_{\eta_i}$, which appear at stage m.

$\mathrm{cn}(\Gamma)$

Here we use $\mathrm{cn}(\Gamma)$ to denote the closure of Γ under logical consequence. Then,

$$\phi_{n+1}(\ldots x_{\eta_i} z^j_{\eta_i} \ldots)_{\eta_i \in v_{n+1}, j<r_{n+1}} \in \Gamma$$

$$\text{implies } \phi_n(\ldots x^*_{\eta_i} z^j_{\eta^*_i} \ldots)_{\eta^*_i \in v_n, j<r_n} \in \mathrm{cn}(\Gamma).$$

Our construction will yield by induction, for $m > n$

$$\phi_m(\ldots x_{\eta_i} z^j_{\eta_i} \ldots)_{\eta_i \in v_m, j<r_m} \in \Gamma$$

$$\text{implies } \phi_n(\ldots x^*_{\eta_i}, z^j_{\eta^*_i} \ldots)_{\eta^*_i \in v_m, j<r_n} \in \mathrm{cn}(\Gamma).$$

The Template Theorem

Theorem

Let $\boldsymbol{K}$ be a class of models of T. If there is a Z-compatible $\mathbb{P}$ that is suitable for T then there is an $M \in \boldsymbol{K}$ of cardinality $2^{\aleph_0}$.

Proof of Template Theorem

We construct by induction $\{(v_n, \phi_n) : n < \omega\} \subset Z$ such that (v_{n+1}, ϕ_{n+1}) is a successor of (v_n, ϕ_n) and $\bigcup_n v_n = 2^{<\omega}$. Γ is the set of ϕ_n. This requires a little bookkeeping; the following technical definitions will be essential.

Let $\chi_i(\mathbf{z})$ be an enumeration of all formulas in variables $\mathbf{z}$ occurring in Y.

Let $\psi_i(\mathbf{w}, u)$ be an enumeration of all formulas in this form.

Induction Step of Construction

We say a formula $\chi_i(\mathbf{z})$ $(\psi_i(\mathbf{w}, u))$ requires attention at stage n of the construction if $i < n$ and the variables of $\mathbf{z}$ (respectively $\mathbf{w}$) occur in $\phi_n(\mathbf{z})$.
At stage n we are given (v_n, ϕ_n) Now by a subinduction on the finite number m of formulas which require attention we modify ϕ_n (actually ϕ_n^k for $k \leq m$ when we consider the kth formula that requires attention) to $\phi'_n = \phi_n^m$.

1. If $\chi_k(\mathbf{z})$ (the *kth* formula requiring attention) is consistent with ϕ_n^k, then ϕ_n^{k+1} is $\phi_n \wedge \chi_i$.
2. If $(\exists u)\psi_k(\mathbf{w}, u)$ is consistent with ϕ_n^k, then choose q by the Henkin witness condition and let ϕ_{n+1}^{k+1} be $\Phi_1(q)$.
3. Now apply the splitting condition.

The Construction works

By this induction we construct a finitely asymptotically similar tree. By compactness we obtain a set of asymptotically similar elements $\boldsymbol{a}_\eta$ of cardinality the continuum. A is completely determined because each extendability condition has been addressed.

Moreover, A is the universe of a model of T since any formula $\psi(\boldsymbol{a}_{\eta_1}, \ldots, \boldsymbol{a}_{\eta_1}, u)$, whose satisfaction is consistent with the diagram of A has a witness in A since we met the Henkin witness conditions.

Section 4:
Application 1: Ackerman, Freer, Patel

Ackerman, Freer, Patel

AFP Theorem

If ψ is a complete sentence of $L_{\omega_1,\omega}$ with totally trivial closure then there is a Borel model of ψ which strongly witnesses ψ. Further, there is an invariant measure concentrated on models of ψ.

We will prove a general theorem which yields the first sentence as one of a family of consequences.

Some notions of closure

Since a countable atomic model M is finitely homogenous the usual notion of algebraic closure, denoted acl ($a \in acl(B)$ iff for some $\phi(x, \mathbf{b})$ has only finitely many for some $\mathbf{b} \in B$

coincides with the Galois or group-theoretic closure, denoted gcl,

($a \in gcl(B)$ iff the orbit of a in $\mathrm{aut}_{\mathbf{b}}\mathrm{M}$ is finite for some $\mathbf{b} \in B$.)

Totally trivial closure

We will say a closure relation is totally trivial if for every set A, $\mathrm{cl}(A) = A$.

Theorem

Let B be a countable model with a totally trivial closure operation. Then there is an uncountable Borel model of $\mathrm{Th}(M)$.

Proof. In this case the $\mathbf{z}_\eta$ in the template are just the $\mathbf{x}_\eta$ with no $\mathbf{r}$ or $\mathbf{y}$. Let $\mathbb{P}$ be the collection of complete formulas in T ordered by implication and let such formulas be the second coordinates of Z. Fix Y exactly as in Definition of Z.
We need to check the suitability conditions. Extendability is trivial as each formula in $\mathbb{P}$ is complete.

Totally trivial closure: Henkin witnesses

Let $(v, \phi(\mathbf{x})_{\eta \in v})) \in Z$, with $\mathbf{w} \subset \mathbf{x} = \langle \mathbf{x}_{\eta_1}, \ldots \mathbf{x}_{\eta_k} \rangle$. Since the closure is totally trivial if $\exists u \psi(\mathbf{w}, u)$ is consistent with $\phi(\mathbf{x})$, a witness for $\psi(\mathbf{w}, z)$ is either found in $\mathbf{x}$ or adding it to $\mathbf{x}$ still gives an independent set.

In the first case do nothing; in the second choose some $\nu \in v$ and extend $\mathbf{x}_\nu$, which has length k, by adding a new element x_ν^{k+1} satisfying $\psi'(\mathbf{x}, u)$ (and so the extended $\mathbf{x}$ is an independent sequence).

Totally trivial closure: proof of splitting

Moreover, we can split $(v, \phi(\mathbf{x}))$ at any $\eta_k \in v$, since total triviality implies that if $\phi(x_{\eta_1}, \ldots, x_{\eta_k}, \ldots, x_{\eta_n})$ is consistent with the x_{η_i} distinct so is $\theta = \phi(x_{\eta_1}, \ldots, x_{\eta_k\frown 0}, \ldots, x_{\eta_n}) \wedge \phi(x_{\eta_1}, \ldots, x_{\eta_k\frown 1}, \ldots, x_{\eta_n})$ along with assertion that all these variable are distinct.

(If not we would have that $\phi(x_{\eta_1}, \ldots, x_{\eta_k}, \ldots, x_{\eta_n})$ implies $x_{\eta_k\frown 0} \in \mathrm{cl}(\phi(x_{\eta_1}, \ldots, x_{\eta_{k-1}}, x_{\eta_{k+1}}, \ldots, x_{\eta_n}).)$ Any completion of θ gives the required splitting.

Strong Witnessing

Definitions AFP

The Borel structure (M, m) with measure m witnesses a consistent Henkin formula $\psi(\mathbf{a}, u)$ either $\psi(\mathbf{a}, u)$ is satisfied by a a_i or $\{b : M \models \psi(\mathbf{a}, b)\}$ has positive measure.

M strongly witnesses *T* if for every non-degenerate probability measure on *M* every Henkin formula is witnessed.

Regaining AFP

AFP Theorem

If ψ is a complete sentence of $L_{\omega_1,\omega}$ with totally trivial closure then there is Borel model of ψ which strongly witnesses ψ. Further, there is in an invariant measure concentrated on models of ψ.

Proof. By applying Theorem 7 to the Scott sentence of M, without loss of generality we may assume that M is an atomic model. So we can apply the previous theorem to get a Borel model. Since in our model, each Henkin formula not

satisfied in its parameters is satisfied on an interval (by the definition of asymptotically similar), the model satisfies strong witnessing.

Measures on countable models

Thus one can apply the remainder of the AFP argument to construct an invariant measure concentrated on models of T.

Section 5:
Application 2: Quasiminimal geometries in the continuum

Quasiminimal structures

Definition

The τ-structure M with cardinality $\aleph_1$ is quasi-minimal if every $L_{\omega_1,\omega}$ definable subset of M is countable or cocountable.

Equivalently,
The τ_1-structure M_1 with cardinality $\aleph_1$ is quasi-minimal if every τ_1-definable subset of M_1 is countable or cocountable.

Quasiminimal closure

Definition

$\mathrm{qcl}(a, M)$ is the set of $b \in M$ such that for some formula ϕ, $\phi(a, b)$ and $\phi(a, M)$ is countable.

Definition

A set X is independent if for every $a \in X$, $a \notin \mathrm{qcl}(X - \{a\})$.

Remark

qcl is monotonic and $\mathrm{qcl}(\mathrm{qcl}(A) = \mathrm{qcl}(A)$. But qcl does not in general define a combinatorial geometry on the universe of each model in $\boldsymbol{K}_T$ (even assuming homogeneity).(ItaiTsuboiWakai)

Homogeneous Quasiminimal structures have models in the continuum

Assumption

qcl satisfies exchange and is therefore a combinatorial geometry.
We also assume that the model in $\aleph_1$ is strongly ω-homogeneous.

Theorem

If there is a quasiminimal structure $M \in \boldsymbol{K}_T$, where qcl satisfies exchange on the universe then there is an $N \in \boldsymbol{K}_T$ with cardinality $2^{\aleph_0}$.

Even strong ω-homogeneity gives the countable closure condition and a unique non-quasialgebraic type.

Comments: Homogeneous Quasiminimal structures have models in the continuum

The proof is another application of the main construction.

The novelty is that $\mathbf{z}_\eta = \mathbf{x}_\eta \mathbf{y}_\eta$
where $\hat{\mathbf{x}} = \bigcup_\eta \mathbf{x}_\eta$ is independent
and each $\mathbf{z}^i_\eta \in \mathrm{acl}(\hat{\mathbf{x}})$.

The $\mathbf{y}_\eta$ are the (algebraic) Henkin witnesses.

The quasiminimality is used for splitting.
Note that a basis need not be a set of indiscernibles, so this argument does not give arbitrarily large models.

This result is orthogonal to the recent paper by Bays, Hart, Hyttinen, Kesala which gives arbitrarily large models from weak quasiminimality conditions on class of countable structures.

Section 5:
Application 3: Pseudo-minimal geometries in the continuum

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Pseudo-algebraic Closures

Definition

b is pseudo-algebraic over $\boldsymbol{a}$ in N, if for some M_1 with $\boldsymbol{ab} \in M_1 \prec N$, for every M chosen with $\boldsymbol{a} \in M \prec M_1$, if Mb is atomic, then $\mathbf{b} \in M$. We write $\mathbf{b} \in \mathrm{pcl}(\boldsymbol{a})$.

Definition

1. Let M be an atomic model of T. A possibly incomplete p type over $\boldsymbol{a}$, which is realized in M, is pseudominimal if for any $\mathbf{c} \in M$ realizing p, and any finite $A \subset M$, $\boldsymbol{a} \subset A$, $A\mathbf{bc} \subset M$, with $\mathbf{b} \subseteq \mathrm{pcl}(\mathrm{A}\mathbf{c})$ but $\mathbf{b} \notin \mathrm{pcl}(\mathrm{A})$, if $\mathbf{c}$ realizes p then $\mathbf{c} \subset \mathrm{pcl}(\mathrm{A}\mathbf{b})$.
2. T is pseudominimal if $x = x$ is pseudominimal.

Two Theorems

A Theorem

If $\boldsymbol{K}_T$ 'fails density of pseudo-minimal types' then $\boldsymbol{K}_T$ has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

We will consider a special case of 'not failing density of pseudo-minimal types': pcl satisfies exchange on the universe.

A Lemma for splitting

Let N be an atomic model. If $\mathbf{b}$ is not pseudo-algebraic over $\boldsymbol{a}$ in N then $\mathrm{tp}(\mathbf{b}/\boldsymbol{a})$ is realized in $N - \mathrm{pcl}(\boldsymbol{a}\mathbf{b})$.

Pseudominimal structures have models in the continuum

Theorem

If there is a pseudominimal $M \in \boldsymbol{K}_T$, where acl satisfies exchange, with cardinality $\aleph_1$, then there is an $N \in \boldsymbol{K}_T$ with cardinality $2^{\aleph_0}$

The construction is similar to the one for quasiminimal. But the unique non-quasialgebraic type is replaced by the 'splitting lemma' from last side.

The **r** is finally needed to show the result for 'almost pseudominimal theories'.

Distinguishing the closure notions

example

Suppose that an atomic model M consists of two sorts. The U-part is countable, but non-extendible (e.g., U infinite, and has a successor function S on it, in which every element has a unique predecessor). On the other sort, V is an infinite set with no structure (hence arb large atomic models). Then, if an element $x_0 \in U$ is not algebraic over $\emptyset$ in the normal sense but is in $\mathrm{pcl}(\emptyset)$.

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Section 7: Further Applications

Further Applications

1. two cardinal models
2. 'diverse classes' many models in non-ω-stable theories
3. Shelah 522: incomplete sentences in $L_{\omega_1,\omega}$ with further combinatorial hypotheses.