Sample solution

page 119, problem 21.i Show f is injective iff $\overrightarrow{f}$ is injective iff $\overleftarrow{f}$is surjective.

Lemma 1 *f is injective implies $\overrightarrow{f}$ is injective.*

Proof. Suppose not. Then there exists $A \neq B \in P(X)$ with $\overrightarrow{f}(A) = \overrightarrow{f}(B)$. Choose $x \in A - B$ or there exists $x \in B - A$. Since the proof for the two cases is identical, just reversing the roles of A and B, we consider only $x \in A - B$. Then $f(x) \in \overrightarrow{f}(A) = \overrightarrow{f}(B)$. So for some $b \in B$, $f(b) = f(x)$ (by definition of $\overrightarrow{f}$). Since $x \notin B$ and $b \in B$, $x \neq b$. But then x and b contradict that f is injective. $\Box_1$

Lemma 2 *$\overrightarrow{f}$ is injective implies $\overleftarrow{f}$ is surjective.*

We must show that for any $A \subset X$, there is a $B \subset Y$ with $\overleftarrow{f}(B) = A$. The obvious candidate is $B = \overrightarrow{f}(A)$. Clearly, $A \subseteq \overleftarrow{f}(B)$, but in general $\overleftarrow{f}(B)$ might be strictly larger than A.

Claim 3 *: For any function f, $\overrightarrow{f}(\overleftarrow{f}(\overrightarrow{f}(A)) = \overrightarrow{f}(A)$.*

proof of claim. If $c \in \overrightarrow{f}(\overleftarrow{f}(\overrightarrow{f}(A))$ then there is an $a \in \overleftarrow{f}(\overrightarrow{f}(A)$ with $f(a) = c$. But $a \in \overleftarrow{f}(\overrightarrow{f}(A)$ means $f(a) \in \overrightarrow{f}(A)$ (by definition of $\overrightarrow{f}$). So $c \in \overrightarrow{f}(A)$.

Conversely, suppose $c \in \overrightarrow{f}(A)$. So $c = f(a)$ for some a that is in $A \subset \overleftarrow{f}(\overrightarrow{f}(A)$ and so $c \in \overrightarrow{f}(\overleftarrow{f}(\overrightarrow{f}(A))$. $\Box_3$.

Now we return to the proof of Lemma 2 and use the fact that f is injective. If $A \neq \overleftarrow{f}(\overrightarrow{f}(A))$ we have two distinct sets mapped to $\overrightarrow{f}(A)$ by $\overrightarrow{f}$; that contradicts the hypothesis that $\overrightarrow{f}$ is injective so we finish the proof that $\overrightarrow{f}$ injective implies $\overleftarrow{f}$ is surjective. $\Box_2$

Lemma 4 *$\overleftarrow{f}$ is surjective implies f is injective.*

If f is not injective, there exist $a \neq b \in X$ with $f(a) = f(b)$. Call this common image c and let $C = \{c\}$ and let $A = \{a\}$. I claim there is no set $D \subset Y$ with $\overleftarrow{f}(D) = A$. As, if $\overleftarrow{f}(D) = A$, c must be in D. But then $\overleftarrow{f}(C) \subseteq \overleftarrow{f}(D)$ so $\{a, c\} \subset \overleftarrow{f}(D)$. So $\overleftarrow{f}(D) \neq A$ since A has only one element. $\square_4$

We are asked to prove the equivalence of three statements. We proved the first implies the second, the second implies the third, and the third implies the first. By the transitivity of implication we finish.