Math 300 Writing for Mathematics Spring 2003

Mathematics for Essay 2

Consider the infinite series:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots+ \qquad (1)$$

This is the case, $p=2$ of the p-series:

$$\Sigma_{n=1}^{\infty}\frac{1}{n^p}.$$

We know from our discussion in Essay 1 that

$$\Sigma_{n=1}^{\infty}\frac{1}{n^2}<2.$$

We noticed in Essay 1 that these are the areas of a collection of squares that fit in a 1 by 2 rectangle with a lot left over. We would like to get a better approximation to the actual value.

Sketch the graph of the function $f(x)=\frac{1}{x^2}$. Note that there are disjoint rectangles with area $\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$ contained in the region bounded by the x-axis, $f(x)=\frac{1}{x^2}$, and the line $x=1$. (Take the rectangle formed by going one unit to the left from the point $(n, \frac{1}{n^2})$ to get area $\frac{1}{n^2}$.) So

$$\int_1^{\infty}\frac{1}{x^2}\,dx>\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots.$$

More generally

$$\int_m^{\infty}\frac{1}{x^2}\,dx>\frac{1}{(m+1)^2}+\frac{1}{(m+2)^2}+\frac{1}{(m+3)^2}+\cdots.$$

So for each m, we can get an upper bound on

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+ \qquad (2)$$

as $1+\frac{1}{4}+\frac{1}{9}+\cdots\frac{1}{m^2}+\int_m^{\infty}\frac{1}{x^2}\,dx$.

The first of these term is finite; find it with a calculator. On the other hand

$$\int_m^{\infty}\frac{1}{x^2}\,dx=\left.\frac{-1}{x}\right|_m^{\infty}=\lim_{b\to\infty}\left.\frac{-1}{x}\right|_m^{b}=\lim_{b\to\infty}(\frac{-1}{b}-\frac{-1}{m})=\frac{1}{m}.$$

For example, taking $m=2$, my estimate is $(1+1/4)+1/2=1\,3/4$, for $m=3$ it is $(1+1/4+1/9)+1/3$ which is approximately 1.69.

Now to estimate the volumes repeat this process but with $g(x)=\frac{1}{x^3}$ replacing $f(x)$. Remarkably, the exact value for $\Sigma_{n=1}^{\infty}\frac{1}{n^2}$ is $\frac{\pi^2}{6}$ while the exact value of $\Sigma_{n=1}^{\infty}\frac{1}{n^3}$ is unknown:

There is an explanation at the following website. But this is not required or expected to be read. http://plus.maths.org.uk/issue19/features/infseries/2pdf/index.html/op.pdf