# Lecture 12: Excellence implies stability transfers

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

October 22, 2003

In this section we consider $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ to be the class of atomic models of a first order theory which is L-excellent.

Our goal is to show that if $\boldsymbol{K}$ is $\omega$-stable then it is stable in all cardinalities.

The crucial point is that if $M = \bigcup_{i<\alpha} M_i$, $p \neq q \in S_{\rm at}(M)$ then for some $i < \alpha$, $p \upharpoonright M_i \neq q \upharpoonright M_i$, because the types are syntactic.

Acknowlegement: This proof is virtually a copy of the first order case; the actual write-up is pirated from a more general version being developed by Baldwin-Kueker-Vandieren.

**Theorem 1** *If $\boldsymbol{K}$ is excellent and $\aleph_0$-stable then $\boldsymbol{K}$ is stable in all $\kappa$.*

Proof. We show that if $\boldsymbol{K}$ in every cardinality less than $\kappa$, then it is $\kappa$-stable.

Take any $M$ of cardinality $\kappa$. We may write $M$ as the union of a continuous chain $\langle M_i \mid i < \kappa \rangle$ under $\prec_{\boldsymbol{K}}$ of models of cardinality $< \kappa$ in $\boldsymbol{K}$.

We say that a type over $M_i$ has many extensions to mean that it has $> \kappa$ distinct extensions to a type over $M$.

**Claim 2** *For every $i$, there is some type over $M_i$ with many extensions.*

Proof. Each type over $M^*$ is the extension of some type over $M_i$ and, by our assumption, there are less than $\kappa$ many types over $M_i$, so at least one of them must have many extensions.

**Claim 3** *For every $i$, if the type $p$ over $M_i$ has many extensions, then for every $j > i$, $p$ has an extension to a type $p'$ over $M_j$ with many extensions.*

Proof. Every extension of $p$ to a type over $M^*$ is the extension of some extension of $p$ to a type over $M_j$. By our assumption there are less than $\kappa$ many such extensions to a type over $M_j$, so at least one of them must have many extensions.

**Claim 4** *For every $i$, if the type $p$ over $M_i$ has many extensions, then for all sufficiently large $j > i$, $p$ can be extended to two types over $M_j$ each having many extensions.*

Proof. By Claim 3 it suffices to establish the result for some $j > i$. So assume that there is no $j > i$ such that $p$ has two extensions to types over $M_j$ each having many extensions. Then, by Claim 3 again, for every $j > i$, $p$ has a unique extension to a type $p_j$ over $M_j$ with many extensions. Let $S^*$ be the set of all extensions of $p$ to a type over $M^*$ – so $|S^*| \geq \kappa^+$. Then $S^*$ is the union of $S_0$ and $S_1$, where $S_0$ is the set of all $q$ in $S^*$ such that $p_j < q$ for all $j > i$, and $S_1$ is the set of all $q$ in $S^*$ such that $q$ does not extend $p_j$ for some $j > i$. Now if $q_1$ and $q_2$ are different types in $S^*$ then, since types are syntactic their restrictions to some $M_i$ must differ. Hence their restrictions to all sufficiently large $M_j$ must differ. Therefore, $S_0$ contains at most one type. On the other hand, if $q$ is in $S_1$ then, for some $j > i$, $q \upharpoonright M_j$ is an extension of $p$ to a type over $M_j$ which is different from $p_j$, hence has at most $\kappa$ extensions to a type over $M^*$. Since there are $< \kappa$ types over each $M_j$ (by assumption) and just $\kappa$ models $M_j$ there can be at most $\kappa$ types in $S_1$. Thus $S^*$ contains at most $\kappa^+$ types, a contradiction.

**Claim 5** *There is a countable $M \prec_{\boldsymbol{K}} M^*$ such that there are $2^{\aleph_0}$ types over $M$.*

Proof. Let $p$ be a type over $M_0$ with many extensions. By Claim 4 there is a $j_1 > 0$ such that $p$ has two extensions $p_0, p_1$ to types over $M_{j_1}$ with many extensions. Iterating this construction we find a sequence of countable models $M_{j_n}$ and a tree $p_s$ of types for $s \in 2^\omega$ with the $2^n$ types $p_s$ (where $s$ has length $n$) all over $M_{j_n}$ and each $p_s$ has many extensions. Let $\hat{M}$ be the union of the $M_{j_n}$. Now for each $\sigma \in 2^\omega$, $p_\sigma = \bigcup_{s \subset \sigma} p_s$ is, by type realizability, in $S_{\rm at}(\hat{M})$ contradicting $\omega$-stability.

This concludes the proof of Theorem 1.