Lecture 6: Galois types and saturation

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We work in this section under the following strong assumption.

Assumption 1 *1. $\boldsymbol{K}$ has arbitrarily large models.*

2. $\boldsymbol{K}$ satisfies the amalgamation property and the joint embedding property.

3. The Lowenheim-number of $\boldsymbol{K}$, $\mathrm{LS}(\boldsymbol{K})$, is $\aleph_0$.

We say $\boldsymbol{K}$ has the amalgamation property if $M \leq N_1$ and $M \leq N_2 \in K$ with all three in $\boldsymbol{K}$ implies there is a common strong extension N_3 completing the diagram. Joint embedding means any two members of $\boldsymbol{K}$ have a common strong extension. Crucially, we amalgamate only over members of $\boldsymbol{K}$; this distinguishes this context from the context of homogeneous structures.

In this section we take advantage of joint embedding and amalgamation to find a monster model. We then define types in terms of orbits of stabilizers of submodels. This allows an identification of 'model-homogeneous' with 'saturated'. That is, we give an abstract account of Morley-Vaught [**?**].

Definition 2 *M is μ-model homogenous if for every $N \prec_{\boldsymbol{K}} M$ and every $N' \in \boldsymbol{K}$ with $|N'| < \mu$ and $N \prec_{\boldsymbol{K}} N'$ there is a $\boldsymbol{K}$-embedding of N' into M over N.*

To emphasize, this differs from the homogeneous context because the N must be *in* $\boldsymbol{K}$. It is easy to show:

Lemma 3 *If M_1 and M_2 are μ-model homogeneous of cardinality $\mu > \mathrm{LS}(\boldsymbol{K})$ then $M_1 \approx M_2$.*

Proof. If M_1 and M_2 have a common submodel N of cardinality $< \mu$, this is an easy back and forth. Now suppose N_1, (N_2) is a small model of M_1, (M_2) respectively. By the joint embedding property there is a small common extension N of N_1, N_2 and by model homogeneity N is embedded in both M_1 and M_2. $\square_3$

Note that in the absence of joint embedding to get uniqueness, we would (as in [**?**]) have to add to the definition of 'M is model homogeneous' that all models of cardinality $< \mu$ are embedded in M.

Exercise 4 *Suppose M is μ-model homogeneous with cardinality μ, $N_0, N_1, N_2 \in \boldsymbol{K}$ with $N_0 \prec N_1, N_2 \prec M$, and f is isomorphism between N_1 and N_2 over N_0. Then f extends to an automorphism of M.*

Theorem 5 *If $\mu *^{<\mu *} = \mu *$ and $\mu * \geq 2^{\mathrm{LS}(\boldsymbol{K})}$ then there is a model $\mathbb{M}$ of cardinality $\mu *$ which is model homogeneous.*

We call the model constructed in Theorem 5, the monster model. From now on all, structures considered are substructures of $\mathbb{M}$ with cardinality $< \mu *$. The standard arguments for the use of a monster model in first order model theory ([**?**, **?**] apply here.

Definition 6 *Let $M \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M ($\in \mathbb{M}$) is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.*

We freely use the phrase, 'Galois type of a over M', dropping the ($\in \mathbb{M}$) since $\mathbb{M}$ is fixed. Note that *a priori* this notion depends on the embedding of Ma into an $N \in \boldsymbol{K}$ and the embedding of N into $\mathcal{M}$. Since we have assumed amalgamation, our usage is justified as long as the base is an $M \in \boldsymbol{K}$. In more general situations, the Galois type is an equivalence class of an equivalence relation on triples (M, a, N). This is an equivalence relation on the class of M that are amalgamations for extensions in the same cardinality. (See [**?**, **?**].) Since we have amalgamation and have fixed $\mathcal{M}$, we don't need the extra notation. The following definition and exercise show the connection of the situation as described here with the more complicated description elsewhere. They are needed only to link with the literature.

Definition 7 *For $M \prec_{\boldsymbol{K}} N_1 \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} N_2 \in \boldsymbol{K}$ and $a \in N_1 - M$, $b \in N_2 - M$, write $(M, a, N_1) \sim (M, b, N_2)$ if there exist strong embeddings f_1, f_2 of N_1, N_2 into some N^* which agree on M and with $f_1(a) = f_2(b)$.*

Exercise 8 *If $\boldsymbol{K}$ has amalgamation, $\sim$ is an equivalence relation.*

Exercise 9 *Suppose $\boldsymbol{K}$ has amalgamation and joint embedding. Show $(M, a, N_1) \sim (M, b, N_2)$ if and only if there are embeddings g_1 and g_2 of N_1, N_2 into $\mathbb{M}$ that agree on M and such that $g_1(a)$ and $g_2(b)$ have the same Galois type over $g_1(M)$.*

Definition 10 *The set of Galois types over M is denoted* $\mathrm{ga} - \mathrm{S(M)}$.

We say a Galois type p over M is realized in N with $M \prec_{\boldsymbol{K}} N \prec_{\boldsymbol{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

Definition 11 *The model M is μ-Galois saturated if for every $N \prec_{\boldsymbol{K}} M$ with $|N| < \mu$ and every Galois type p over N, p is realized in M.*

Again, *a priori* this notion depend on the embedding of M into $\mathcal{M}$; but with amalgamation it is well-defined.

The following model-homogeneity=saturativity theorem was announced with an incomplete proof in [**?**]. Full proofs are given in Theorem 6.7 of [**?**] and .26 of [**?**]. However, we give a simpler argument.

Theorem 12 *For $\lambda > \mathrm{LS}(\boldsymbol{K})$, The model M is λ-Galois saturated if and only if it is λ-model homogeneous.*

Proof. It is obvious that λ-model homogeneous implies λ-Galois saturated. It is easy to prove the converse by induction on cardinality if one has the successor stage. So we assume $|M| = \mu^+$, and $M \prec_{\boldsymbol{K}} \mathbb{M}$ is μ^+-saturated. We want to show M is μ^+-model homogeneous. So fix $M_0 \prec_{\boldsymbol{K}} M$ and N with $|N| = \mu$ and $M \prec_{\boldsymbol{K}} N \prec_{\boldsymbol{K}} \mathbb{M}$. We must construct an embedding of N into M. Enumerate $N - M$ as $\langle a_i : i < \mu \rangle$. We will define f_i for

$i < \mu$ an increasing continuous sequence of maps with domain N_i and range M_i so that $M_0 \prec_{\boldsymbol{K}} N_i \prec_{\boldsymbol{K}} \mathbb{M}$, $M_0 \prec_{\boldsymbol{K}} M_i \prec_{\boldsymbol{K}} M$ and $a_i \in N_{i+1}$. The restriction of $\bigcup_{i<\mu} f_i$ to N is required embedding. Let $N_0 = M_0$ and f_0 the identity. Suppose f_i has been defined. Choose the least j such that $a_j \in N - N_i$. By the model homogeneity of $\mathbb{M}$, f_i extends to an automorphism $\hat{f}_i$ of $\mathbb{M}$. Using the saturation, let $b_j \in M$ realize the Galois type of $\hat{f}_i(a_j)$ over M_i. So there is an $\alpha \in \operatorname{aut}\mathbb{M}$ which fixes M_i and takes b_j to $\hat{f}_i(a_j)$. Choose $M_{i+1} \prec_{\boldsymbol{K}} M$ with cardinality μ and containing $M_i b_j$. Now $\hat{f}_i^{-1} \circ \alpha$ maps M_i to N_i and b_j to a_j. Let $N_{i+1} = \hat{f}_i^{-1} \circ \alpha(M_{i+1})$ and define f_{i+1} as the restriction of $\alpha^{-1} \circ \hat{f}_i$ to N_{i+1}. Then f_{i+1} is as required. $\square_{12}$

In the remainder of this section we discuss some important ways in which Galois types behave differently from 'syntactic types'.

Note that if $M \prec_{\boldsymbol{K}} N \prec_{\boldsymbol{K}} \mathbb{M}$, then $p \in \mathrm{ga-S(N)}$ extends $p' \in \mathrm{ga-S(N)}$ if for some (any) a realizing p and some (any) b realizing p' there is an automorphism α fixing M and taking a to b.

Lemma 13 *If $M = \bigcup_{i<\omega} M_i$ in an increasing chain of members of $\boldsymbol{K}$ and $\{p_i : i < \omega\}$ satisfies $p_{i+1} \restriction M_i = p_i$, there is a $p_\omega \in \mathrm{ga-S(M)}$ with $p_\omega \restriction M_i = p_i$ for each i.*

Proof. Let a_i realize p_i. By hypothesis, for each $i < \omega$, there exists f_i which fixes M_{i-1} and maps a_i to a_{i-1}. Let g_i be the composition $f_0 \circ f_1 \circ \ldots f_i$. Then g_i maps a_i to a_0, fixes M_0 and $g_i \restriction M_{i-1} = g_{i-1} \restriction M_{i-1}$. Let M_i' denote $g_i(M_i)$ and M' their union. Then $\bigcup_{i<\omega} g_i$ is an isomorphism between M and M'. So by model-homogeneity there exists an automorphism h of $\mathbb{M}$ with $h \restriction M_i = g_i \restriction M_i$ for each i. Now $g_i^{-1} \circ h$ fixes M_i and maps a_ω to a_i for each i. This completes the proof. $\square_{13}$

Now suppose we wanted to prove Lemma 13 for chains of length $\delta > \omega$. The difficulty can be seen at stage ω. In addition to the assumptions of Lemma 13, we are given $\{a_i : i \leq \omega\}$ and $f_{\omega,i}$ which fixes M_i and maps a_ω to a_i. We can construct g_i as in the original proof. The difficulty is to find g_ω which extends all the g_i and maps a_ω to a_0. In the argument for Lemma 13, we found a map h and an element (which we will now call a'_ω such that h takes a'_ω to a_0 while h extends all the g_i. We would be done if a_ω and a'_ω realized the same galois type over $M = M_\omega$. In fact, a_ω and a'_ω realized the same galois type over each M_i. So the following *locality* condition (for chains of length ω) would suffice for this special case. Moreover, by a further induction locality would give Lemma 13 for chains of arbitrary length. Unfortunately, locality probably does not hold for all AEC with amalgamation.

Definition 14 *$\boldsymbol{K}$ has local galois types if for every $M = \bigcup_{i<\kappa} M_i$ in a continuous increasing chain of members of $\boldsymbol{K}$ and for any $p, q \in \mathrm{ga-S(M)}$: if $p \restriction M_i = q \restriction M_i$ for every i then $p = q$.*

We have sketched the proof of:

Lemma 15 *Suppose $\boldsymbol{K}$ has local Galois types. If $M = \bigcup_{i<\kappa} M_i$ in an increasing chain of members of $\boldsymbol{K}$ and $\{p_i : i < \kappa\}$ satisfies $p_{i+1} \restriction M_i = p_i$, there is a $p_\kappa \in \mathrm{ga-S(M)}$ with $p_\kappa \restriction M_i = p_i$ for each i.*

Locality provides a key distinction between the general AEC case and homogenous structures. In homogeneous structures, types are syntactic objects and locality is trivial. Thus, as pointed out by Shelah, Hyttinen, and Buechler-Lessmann, Lemma 15 applies in the homogeneous context.