Lecture 8: Morley's method for Galois Types: Downward categoricity

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Now we prove 'Morley's method' for Galois types.

Lemma 1 *[II.1.5 of 394] If $M_0 \leq M$ and M is huge we can find an EM-set Φ such that the following hold.*

1. *The τ-reduct of the Skolem closure of the empty set is M_0.*
2. *For every I, $M_0 \leq EM(I, \Phi)$.*
3. *If I is finite, $EM_\tau(I, \Phi)$ can be embedded in M.*
4. *$EM_\tau(I, \Phi)$ omits every galois type over N which is omitted in M.*

Proof. Let τ_1 be the Skolem language given by the presentation theorem and consider M as the reduct of τ_1 structure M^1. Add constants for M_0 to form τ_1'. Now apply Lemma **??** to find an EM-diagram Φ (in τ_1') with all τ-types of finite subsets of the indiscernible sequence realized in M. Now 1) and 2) are immediate. 3) is easy (using clause 5) of Theorem **??** since we chose Φ so all finite subsets of the indiscernible set (and so their Skolem closures) are realized in M.

The omission of Galois types is more tricky. Consider both M and $N = EM_\tau(I, \Phi)$ embedded in $\mathbb{M}$. Let N^1 denote the τ_1'-structure $EM(I, \Phi)$. We need to show that if $a \in N$, $p = \mathrm{ga-tp(a/M_0)}$ is realized in M. For some $\mathbf{e} \in I$, a is in the τ_1-Skolem hull $N_{\mathbf{e}}$ of $\mathbf{e}$. (Recall the notation from the presentation theorem.) By 3) there is an embedding α of $N_{\mathbf{e}}$ into M^1 over M_0. α is also an isomorphism of $N_{\mathbf{e}} \restriction \tau$ into M. Now, by the model homogeneity, α extends to an automorphism of $\mathbb{M}$ fixing M_0 and $\alpha(a) \in M$ realizes p. $\square_1$

This has immediate applications in the direction of transferring categoricity.

Theorem 2 *Suppose $M \in \boldsymbol{K}$ omits a Galois type p over a submodel M_0 with $|M| \geq \mu(|M_0|)$. Then there is no regular cardinal $\lambda \geq |M|$ in which $\boldsymbol{K}$ is categorical.*

Proof. By Lemma 1, there is a model $N \in \boldsymbol{K}$ with cardinality λ which omits p. But by Lemma **??**, the unique model of power λ is saturated. $\square_2$

In [**?**] Shelah asserts the following result:

Theorem 3 *If $\boldsymbol{K}$ is categorical in a regular cardinal λ and $\lambda > \mu(\mu(|\tau|))$ then $\boldsymbol{K}$ is categorical in every θ with $\mu(|\tau|) \leq \theta \leq \lambda$.*

Here is a sketch of the argument. We have shown that there are saturated models of power θ for every $\theta < \lambda$. The obstacle to deducing downward categoricity is that Theorem 1 only allows us to transfer the omission of types when the model omitting the type is much bigger than the domain of the type. The first step in remedying this problem is to show that all types are determined by 'relatively small' subtypes. More precisely, we need the notion that Grossberg and Van Dieren [**?**] have called χ-tame and Shelah [**?**] refers to has 'having χ-character'. We add an extra parameter to be careful.

Definition 4 *We say* $\boldsymbol{K}$ *is* (χ, μ)*-tame if for any saturated* $N \in \boldsymbol{K}$ *with* $|N| = \mu < \lambda$ *if* $p, q, \in \mathrm{ga-S(N)}$ *and for every* $N_0 \leq N$ *with* $|N_0| \leq \chi$*,* $p \upharpoonright N_0 = q \upharpoonright N_0$ *then* $q = p$*.*

Shelah asserts the following in Sections II.1 and II.2.3 of the published version of [**?**]. The published proof is incomplete; I haven't yet seen the corrections. But it seems to use only Ehrenfeucht-Mostowski type methods.

Theorem 5 *Suppose* $\boldsymbol{K}$ *is* λ*-categorical for* $\lambda \geq \mu(\tau)$ *and* λ *is regular. Then* $\boldsymbol{K}$ *is* (χ, χ_1)*-tame for some* χ *and any* χ_1 *with* $\chi < \mu(\tau) \leq \chi_1 \leq \lambda$*.*

The naive argument would give $\chi = \mu(\tau)$ since one is omitting types. But omitting in every cardinal below $\mu(\tau)$ is as good as in $\mu(\tau)$ so the conclusion becomes for some χ with $\chi < \mu(\tau)$.