Formal Language of Geometry

Connection axioms

labeling

angles and congruence

Birkhoff-Moise

A formalization of Hilbert's Axioms

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Quiz

1. Suppose two mirrors are hinged at 90^o. Are the following two statements equivalent.
    1. No matter what angle you look at the mirror you will see your reflection.
    2. A line incident on one mirror is parallel to the line reflected from the second.
2. What is a proof?
3. In teaching experiment 19 (Harel and Souder), what was the student's misconception about a line?

Goals

Formal languages arose to remedy the lack of precision in natural language.

1. Understand the use of axiom systems to 'define' fundamental notions.
2. See exact formalization of Hilbert as a basis for independence proofs.

Geometric Goals

Over the course of the semester we want to understand three geometric assertions and why we might or might not accept them.

1. vertical angles are equal
2. the congruence theorems
3. the parallel postulate

Outline

1 Formal Language of Geometry

2 Connection axioms

3 labeling

4 angles and congruence

5 Birkhoff-Moise

Plane Geometry

We are modifying Hilbert's axioms in several ways. Numbering is as in Hilbert.

We are only trying to axiomatize plane geometry so anything relating to higher dimensions is ignored.

Note difference from Weinzweig.

The objects

Two Sorts

There are two unary predicates: P and L.

$$(\forall x)P(x) \vee L(x).$$

$$(\forall x)P(x) \rightarrow \neg L(x).$$

Relations

binary relation: $I(x, y)$ "x on y"
ternary relation: $B(a, b, c)$ "(c between a and b)"

Connection Axioms

$$(\forall x)(\forall y) I(x, y) \rightarrow P(x) \wedge L(y).$$

There is a unique line between any two points.
exists

$$(\forall x)(\forall y)[P(x) \wedge P(y) \rightarrow (\exists z) L(z) \wedge I(x, z) \wedge I(y, z)].$$

only one

$$(\forall x)(\forall y)(\forall z)(\forall w)([x \neq y \wedge I(x, z) \wedge I(y, z) \wedge I(x, w) \wedge I(y, w)] \rightarrow w = z).$$

Examples

Can we construct finite models for these axioms?

Order Axioms

II.1

$$(\forall x)(\forall y)(\forall z)B(x,y,z) \rightarrow B(y,x,z).$$

II.2 If two points are on a line there is a point on the line between them and a point so that one of these is between the other and the chosen point.

$$(\forall x_1)(\forall x_2)[(\exists w)[\bigwedge_{i \leq 2} I(x_i,w)$$

$$\wedge(\exists z)B(x_1,x_2,z) \wedge (\exists w)B(x_1,w,x_2).$$

Order Axioms II

II.3 If three points are on a line there is always one and only which lies between the other two.

$$(\forall x_1)(\forall x_2)(\forall x_3)[(\exists w)[\bigwedge_{i \leq 3} I(x_i, w)$$

$$\rightarrow B(x_1, x_2, x_3) \vee B(x_2, x_3, x_1) \vee B(x_1, x_3, x_2).$$

II.4 Four points on a straight line can be labeled so that $B(A, D, B), B(A, D, C), B(B, D, B), B(A, C, B)$.

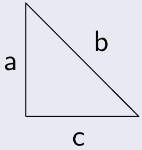
Labeling

Is this a correct statement of the Pythagorean Theorem?

$$a^2 + b^2 = c^2$$

What about?

$c^2 + a^2 = b^2$



Definitions

Segment

If A, B are on a line C is on the segment AB if $B(A, B, C)$.

Congruence

Segment Congruence

Introduce a new 4-ary relation $AB \cong CD$.
Intended meaning $AB \cong CD$ iff AB is congruent to CD.

Congruence

Axioms on Segment Congruence

Congruence is an equivalence relation that is preserved when segments are concatenated.

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Axioms on Segment Congruence

Congruence is an equivalence relation that is preserved when segments are concatenated.

Fix the segment: AB. On any ℓ through any C on ℓ, there are two points D, D' on ℓ with $CD \cong AB$, $CD' \cong AB$.
Hilbert (IV.1, IV.2, IV.3)

Taking Stock

Toward angle congruence

Need to define "angle" carefully.

Pasch's Axiom

Hilbert II.5

A line which intersects one edge of a triangle and misses the three vertices must intersect one of the other two edges.

Another Definition

Halfplane

If A, B are on a line, C and D are in the same halfplane AB if the segment CD does not intersect the line determined by AB.

Half-plane Theorem

Theorem

Every line divides the plane into two half-planes.

Finally Angles

ray

Using the betweenness predicate we can define: A and A' are on the same side of O, (when A, A', O are collinear)
All points on ℓ that are the same side O form the ray from 0.

angles

An angle (h, k) is a pair of non-collinear rays from a point C.
Note that (h, k) splits the plane into two connected regions.
(A region is connected if any two points are connected by a polygonal path.)
The region such that any two point are connected by a segment entirely in the region is called the interior of the angle.

Congruence II

Angle Congruence

Introduce a new 4-ary relation $(h, k) \cong (h', k')$.
Intended meaning $(h, k) \cong (h', k')$ iff (h, k) is congruent to (h', k').

Congruence II

Axioms on Angle Congruence

Angle Congruence is an equivalence relation that is preserved when angles are concatenated.

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Axioms on Angle Congruence

Angle Congruence is an equivalence relation that is preserved when angles are concatenated.

Fix the angle: (h, k). Given an ℓ with C on ℓ, there is a line h' through C so that $(h, k) \cong (h', \ell)$.
Hilbert (IV.4)

Axiom on Triangle Congruence

SAS' AXIOM

If two sides and the included angle are congruent so are the other two angles.

SAS' Theorem

If corresponding angles and two corresponding sides are congruent the third pair of corresponding sides are congruent.

Embedding the reals

The formal structure for books in the SMSG style.

The objects

Three Sorts

There are three unary predicates: P, R and L.

$$(\forall x)P(x) \vee L(x) \vee R(x).$$

Axioms saying P, L and R are disjoint, e.g.

$$(\forall x)P(x) \rightarrow \neg L(x).$$

$$(\forall x)P(x) \rightarrow \neg R(x).$$

Relations

binary relation: $I(x, y)$ "x on y"

functions

Binary functions $+, \cdot$ are defined on R and constant symbols 0, 1.

d maps $P \times P$ to R

Axioms

1. All axioms for the real field.
2. The ruler postulate.
   For any real number r there exist points A, B so that $d(A, B) = r$.
3. The protractor postulate.
4. Geometric axioms (see particular book).