Solutions to Assignment 10: due Nov. 12

John T. Baldwin

November 30, 2007

Redoing Descartes

We now go in the opposite direction from the last two weeks. You are given the real field and may assume anything you know about this structure. The goal is to find a model of Hilbert's axioms.

Definition 1 *1. The set of points will be $R \times R$.*

2. *A line is formally a pair (b, m), where b is a real number and m is either a real number or the symbol ∞.*

3. *The point (x, y) is incident to*

   (a) *(m, b) if $y = mx + b$ where m and b are both real or*

   (b) *(b, ∞) if $x = b$*

1. Show that in the model we have just defined the axioms I.1, I.2, I.3 of Hilbert are true.

Solution: I check only I.1 and I.3.

Postulate I.1. For every two points A, B there exists a line a that contains each of the points A, B.

Use two point form to find the equation of the line through A, B. If they have the same x coordinate a_1, the line is (a_1, ∞).

Postulate I.3 There exists at least two points on a line. There exist at least three points that do not lie on a line.

Let $y = mx + b$. For any real number a, $(a, ma + b)$ is on the line (is incident to) so the line has infinitely many points.

For any two points $(a_1, a_2), (b_1, b_2)$, the line through them is given by $\frac{y-a_2}{x-a_1} = \frac{b_2-a_2}{b_1-a_1}$. Let $m = \frac{b_2-a_2}{b_1-a_1}$ For any a and any c and any non-zero ϵ, the point $(a, a + mc + \epsilon)$ is not on the unique (since you checked I.2) line through $(a_1, a_2), (b_1, b_2)$.

Note that I did not check the vertical line case which is even easier.

2. Define a relation $B((a_1, a_2), (b_1, b_2), (c_1, c_2))$ to interpret $\mathbf{c}$ is between $\boldsymbol{a}$ and $\mathbf{b}$. Check that your definition verifies axiom group II of Hilbert in the model.

Solution: Definition of B

We must define algebraically a relation B on triples $(a_1, a_2), (b_1, b_2), (c_1, c_2)$ that is satisfied if and only if (c_1, c_2) is on the line segment between (a_1, a_2), and (b_1, b_2).

Case 1: $a_1 \neq b_1$.

The line is given by $\frac{y-a_2}{x-a_1} = \frac{b_2-a_2}{b_1-a_1}$.

That is, if I set $m = \frac{b_2-a_2}{b_1-a_1}$, the equation of the line is

$$y = mx + a_2 - ma_1.$$

and (c_1, c_2) is on the segment if $c_2 = mc_1 + a_2 - ma_1$ (that is (c_1, c_2) is on the line and c_1 is between a_1 and b_1). (Formally, either $a_1 \leq c_1 \leq b_1$ or $b_1 \leq c_1 \leq a_1$ depending on which of a_1 and b_1 is greater.)

case 2: $a_1 = b_1$.

The line is (a_1, ∞) and we must have c_2 between a_2 and b_2.

end of solution definition of betweeness.

Now of course as in slide 15 of the lecture: Formalization of Axioms we can define the segment between any two points.

Verify some of the order (betweeness) axioms

Remember that below three points collinear means there is a single linear equation satisfied by all three of them.

Only Pasch is really hard. I am writing this out in far far more detail than I expected from the class. I said you could use any algebraic facts. I am reducing (for at least some of the many cases), these 'facts' to the axioms of ordered fields.

I hadn't intended this degree of complication; it just shows how much hidden algebra is buried in a metric approach to geometry.

We need to check algebraically the picture that if ℓ_1 and ℓ_2 intersect and the slope of ℓ_1 is less than the slope of ℓ_2 then to the right of the point of intersection ℓ_2 is above ℓ_1. Formally,

Fact 2 *If $y = m_1x + b_1$ and $y = m_2x + b_2$ are two lines with (a_1, a_2) on both and $0 < m_1 < m_2$ then for every $x > a_1$ $m_1x + b_1 < y = m_2x + b_2$.*

Proof, $m_1a_1 + b_1 = m_2a_1 + b_2$ so $(m_2 - m_1)a_1 = b_2 - b_1 > 0$ and $(m_2 - m_1)x$ is an increasing function (since $m > 0$ and $x_1 < x_2$ implies $mx_1 < mx_2$; we are assuming any basic algebraic fact). So if $x > a_1$, $m_2x > m_1x$ and $b_2 > b_1$ and we finish. $\Box$??

Suppose $(a_1, a_2), (b_1, b_2), (c_1, c_2)$ are three points that are not all collinear and that the line ℓ cuts $(a_1, a_2), (b_1, b_2)$ in (d_1, d_2). We must show that ℓ cuts either the *segment* $(b_1, b_2), (c_1, c_2)$ or the *segment* $(a_1, a_2), (c_1, c_2)$.

Let ℓ_1 be the line through $(a_1, a_2), (b_1, b_2)$ and let the slope-intercept form for ℓ_1 be $y = m_1x + g_1$.

Let ℓ_2 be the line through $(a_1, a_2), (c_1, c_2)$ and let the slope-intercept form for ℓ_2 be $y = m_2x + g_2$.

Let ℓ_3 be the line through $(b_1, b_2), (c_1, c_2)$ and let the slope-intercept form for ℓ_3 be $y = m_3x + g_3$.

Let ℓ be line that intersects ℓ_1 in some point (d_1, d_2) and let the slope-intercept form for ℓ_3 be $y = mx + g$.

There are a number of cases depending on the relationship of the lines. We consider the case: $0 < m_2 < m_1$ and $a_1 < c_1$. This easily implies $m_3 < 0$.

Recall that the line ℓ cuts ℓ_1 (line through $(a_1, a_2), (b_1, b_2)$) in (d_1, d_2) and suppose it intersects ℓ_2 in (e_1, e_2).

By question 1) we know there is some (e_1, e_2) which is the intersection of ℓ and ℓ_2. If $a_1 < e_1 < c_1$, we are finished.

Case 1. $e_1 < a_1$. I argue below this implies $0 < m_2 < m < m_1$. By Fact **??**, to the right of d_1, ℓ is below ℓ_1 and to the right of e_1, ℓ_2 is below ℓ. So ℓ must intersect ℓ_3 on the segment $(b_1, b_2)(c_1, c_2)$.

The remainder of this case is showing the slopes actually behave as I claim: $0 < m_2 < m < m_1$.

Since $e_1 < a_1$,the slope of ℓ_1 is $\frac{d_2-e_2}{d_1-e_1}$. Then the slope m_1 of ℓ_1 is $\frac{d_2-a_2}{d_1-a_1} = \frac{b_2-d_2}{b_1-d_1}$ (since the three points are collinear). Now, I claim

$$\frac{d_2 - e_2}{d_1 - e_1} < \frac{d_2 - a_2}{d_1 - a_1}.$$

To see this note that for the numerators $d_2 - e_2 < d_2 - a_2$ since $e_2 < a_2$ as ℓ_2 has positive slope. And for the denominators $d_1 - e_1 > d_1 - a_1$ since $e_1 < a_1$. That is if m is slope of ℓ, we have shown $m < m_1$. We know ℓ and ℓ_1 intersect at (d_1, d_2). Now for any $x > d_1$, $mx + g < m_1x + g_1$.

Now also $m > m_2$. As,

$$\frac{a_2 - e_2}{a_1 - e_1} < \frac{d_2 - e_2}{d_1 - e_1}.$$

Again, comparing the numerators: $a_2 - e_2 < d_2 - e_2$ since $d_2 < a_2$ and comparing the denominators $a_1 - e_1 > d_1 - e_1$ since $a_1 > d_1$. So for any $x > e_1$, $mx + g > m_2x + g_2$. Thus the intersection of ℓ and ℓ_3 must be on the segment $(b_1, b_2), (c_1, c_2)$.

Case 2) Suppose $e_1 > c_1$. Now we will have $m < 0$ (since $e_2 > d_2$ and $e_1 > d_1$). Now for $d_1 \leq x \leq e_1$: $mx + g < m_1x + g_1$ (since one function is increasing and the other decreasing) and $mx + g > me_1 + e_2 = m_2e_1 + g_2 \geq m_2x + g_2$. So again the intersection of ℓ and ℓ_3 must be on the segment $(b_1, b_2), (c_1, c_2)$.

I have not checked the various cases with negative slopes or if one of the lines is vertical.

3. Prove the parallel postulate (Hilbert III) holds in this model.

Solution:

Let ℓ given by $y = mx + b$ be a line and (a_1, a_2) a point not on ℓ. I claim the line given by $\frac{y-a_2}{x-a_1} = m$ does not intersect ℓ, We can write the equation of this second line as $y = mx + a_2 - ma_1$. If they intersect, we have $a_2 - ma_1 = 0$, contrary to hypothesis.

Thus we have found a line parallel to ℓ through (a_1, a_2) as required.

Definition 3 *Two segments are congruent if the Cartesian distance between their endpoints is the same.*

4. Prove this model satisfies Hilbert's first three congruence axioms: IV.1, IV.2, IV.3.

The most difficult is IV.1. If A, B are two points on a line a, and A' is a point on the same or on another line a' then it is always possible to find a point B' on a given side of the line a' such that AB and A'B' are congruent.

If $A = (a_1, a_2)$ and $B = (b_1, b_2)$, we have $d(A, B) = \sqrt{((a_2-a_1)^2+(b_2-b_1)^2)}$.

The answer I was expecting was:

Let A' be on the line ℓ' with equation $y = m'x + b'$. Choose B' as either of the two points on that line with $d(A', B') = d(A, B)$.

However, to get a really complete answer, one must show algebraically that there are two such points. To see this, note that we need to find x and y so that, setting $D = d(A, B)$ for ease of writing,

$$x^2 + y^2 = D^2$$

and $(a_1 + x, a_2 + y)$ is on ℓ'. That is, $y = m'x$. So we have to solve the equation

$$x'^2 + (m'x)^2 = D^2$$

in the real field, which is easy.