Independence

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Truth and Validity

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Validity

The sentence ϕ is valid if it is true in every structure.

For every M, $M \models \phi$.

Logical Implication

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Γ logically implies ϕ
(written $\Gamma \models \phi$) means

For every M,
If $M \models \gamma$ for each $\gamma \in \Gamma$ then
$M \models \phi$

formal proof

A formal proof from a set of axioms Γ
is a sequence of wff's such that each one

1. is a member of Γ
2. or is a logical axiom
3. or follows from earlier lines by modus ponens.

We write $\Gamma \vdash \phi$ if there is a proof of ϕ from the hypotheses Γ.

But we will just give normal mathematical proofs and suppress the use of logical axioms.

The Extended completeness theorem

Gödel Ia

$\Gamma \vdash \phi$ if and only $\Gamma \models \phi$

Some important sets of axioms

1. axioms for arithmetic
2. Axioms for the real field $(\Re, +, \times, <, = 0, 1)$
3. axioms for set theory
4. axioms for geometry

Independence

The set of sentences Γ is independent if for $\gamma \in \Gamma$,

$$\Gamma - \{\gamma\} \not\vdash \gamma.$$

Proving independence

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This is common sense; formally it follows from the extended completeness theorem.

The compactness Theorem

Gödel Ia

If for every finite $\Gamma_0 \subset \Gamma$, $\Gamma \cup \{\phi\}$ has a model then $\Gamma \cup \{\phi\}$ has a model.

The Axiom of Archimedes

Hilbert V.1

If AB and CD are any segments, then there exists a number n such that n copies of CD constructed contiguously from A along the ray AB will pass beyond the point B.

Infinite elements

Independence of Archimedes Axiom

Let Γ be the axioms of Hilbert's geometry (groups I-IV) +
$\{B(A_0, B, A_n) : n < \omega\}$
$A_0A_1 \cong A_nA_{n+1}$

Every finite subset of Γ is consistent so there is a model of Hilbert's axioms (w/o) the axiom of archimedes that has an infinite element.