

Geometry and Proof: Course Summary

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December 1, 2007

This course is designed to explicate the following motto attributed to Hilbert.

”It must always be possible to substitute ‘table’, ‘chair’ and ‘beer mug’ for ‘point’, ‘line’ and ‘plane’ in a system of geometrical axioms.”

We have considered this theme where ‘geometrical axioms’ are statements in Tarski’s world, various systems of geometry, and axioms for algebraic systems like the real numbers and integers.

A key point is that for any given set of first order axioms for geometry, there are many many models of those axioms.

1 Basic Logic

The following key notions should be known: formal language, vocabulary, sentence, the compositional theory truth, model/structure, truth in a model, validity, consistency, completeness theorem, compactness theorem.

Know the difference between \vdash and \models . Note we first define $M \models \phi$ and then derive the usage $\Gamma \models \phi$.

The major skill is to be able to decide if a particular sentence in a formal language is true in a particular model.

2 Geometry

There are really two different ways of categorizing geometry involved in this discussion.

Synthetic geometry means that some set of axioms are fixed and proofs take place from those axioms. A *metric geometry* has a notion of numerical length in its fundamental vocabulary. A synthetic geometry may or may not be metric. And one can develop metric geometry axiomatically or not.

2.1 Vertical Angles

The ‘fact’ that vertical angle are equal requires some axiom. Euclid assumed essentially ‘all straight angles are equal’. Some modern texts just assume ‘vertical angles’ are equal. But one cannot just read this from the picture.

If one takes a metric approach and assumes the ‘protractor postulate’ – each angle has a measure in degrees, then the property that all straight angles are equal is immediate because, by definition, they measure 180° and all right angles are congruent because they each measure 90° . This is dual to Euclid’s approach, he explicitly assumes any two right angles (recall Definition 10 of right angle) are congruent. Then in Propositions 13 and 14 he essentially proves all straight angles are equal. (I say essentially because for Euclid all angles have less than 180° ; he speaks obliquely of the sum of two angles being the same as the sum of two right angles.)

2.2 Superposition and Congruency

Euclid’s proof of SAS is vitiated by his use of an ill-defined notion of ‘superposition’. This can be resolved in at least two ways. Hilbert assumes the axiom SAS. Other authors, e.g. Weizsaecker, give a formal treatment with transformations of the plane as objects of the geometry and assume an explicit superposition axiom. The Moise-Birkhoff version also assumes SAS.

Note that even to discuss SAS, there must be a formal definition of angle. A major motivation for discussing half-planes is to provide a precise definition of angle [1].

2.3 Area

Problem. What justifies the formula $A = \frac{bh}{2}$ for the area of a triangle?

Definition 1 *Two polygons are said to be of equal area when they can be decomposed into a finite number of triangles which are respectively congruent to one another in pairs.*

Under Definition 1 one can easily see that the area of triangle is one-half the area of a rectangle. If the sides of the rectangle are commensurable then thinking of breaking the base into b units and the height into h units gives the area of the triangle as $\frac{bh}{2}$.

Commensurability is *essential* for this result using Definition 1. In a non-Archimedean geometry (even satisfying the parallel postulate) there are triangles with same base and height that do not have the same area.

But commensurability is not essential if we slightly weaken the notion of area. The distinction between content and area is found in [2]

Definition 2 *Two polygons are said to be of equal content when it is possible, by the addition of other polygons having equal area, to obtain two resulting polygons having equal area.*

We sketched in [10] the proof of the following theorem. It needs only axiom groups I-IV of Hilbert. The parallel postulate is essential to prove that the opposite sides of parallelograms have the same length (which is an important lemma). See the diagram labeled: Euclid’s diagram for area of a triangle or I.35 and I.38 in [3]

Theorem 3 *If two triangles have the same height and the same base then they have the same content.*

Remark 4 The distinction between content and area is very advanced. I would see no problem with a high school course that defined area by Hilbert’s notion of content and then proved the theorem.

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2.4 Arithmetic and Geometry

Proofs of the following two theorems were sketched in [4] and homework assignments 8,9, and 10.

Theorem 5 [Euclid/Hilbert]: *In a geometry satisfying Hilbert's axioms it is possible to define using the vocabulary of geometry, operations \otimes , \ominus , \oplus on the points of a line so that the line with these operations is a field.*

Fine point: Hilbert actually finds the structure of what I (idiosyncratic notation) will call a pseudofield: the elements are equivalence classes of segments under congruence. \oplus is associative, commutative, and $a \oplus b = a \oplus c$ implies $b = c$. But there are no inverses. The multiplication is a commutative group. And the distributive law holds. In [4], I choose a point 0 on a line and take the segment $0A$ as the representative of the equivalence class of segments congruent to $0A$. Then I define an actual field.

Theorem 6 (Descartes) *Given an ordered field (e.g. the reals or rationals), it is possible to define using the arithmetic operations a system of points and lines and relations of incidence, betweenness, and congruence to satisfy Hilbert's axioms.*

Thus one of the theory of geometry and the theory of fields is consistent if and only if the other is.

2.5 Proportionality

We want to understand the following fundamental theorem.

Theorem 7 *Corresponding sides of similar triangles are proportional.*

This is straightforward if the sides are commensurable. See Euclid VI.1 and VI.2 ([10, 3]). (In a nutshell, if two triangles ABD and ABE have the same height, and the bases are commensurable the area of ABD is to the area of ABE as AD is to AE. (Just break them up into smaller triangle. If it takes m unit intervals to make up AB and n unit intervals to make up AD, it will take n triangles to fill ABD and m identical triangles to fill ABE. The transfer to VI.2 is easy in Euclid.)

But if AE and AD are not commensurable, what does proportional mean?

There are three alternative ways to solve the problem of incommensurability.

1) The solution in most modern geometry books is to study *metric* geometry. That is introduce real numbers and the ruler postulate so that one speaks directly of the lengths of the lines. This is an effective way to hide the question of how real numbers are constructed as limits of rationals. It implicitly assumes the Axiom of archimedes and the Dedekind completeness of the real numbers.

2) Euclid following Eudoxus defines what it means for segments to be proportional. Write AB is to CD as EF is to GH as:

$$AB : CD :: EF : GH$$

This proportion is *defined* to hold if for every pair of natural numbers m and n .

If m disjoint copies of AB can be placed in n disjoint copies of EF

if and only if

then m disjoint copies of EF can be placed in n disjoint copies of GH .

This solution depends on assuming the Axiom of Archimedes: for any two segments AB and CD there is a natural number n such that CD is contained in n disjoint copies of AB (or vice versa).

3) Hilbert provides a third solution. As in section 3) define a (pseudo)-field. Using the elements of this pseudo-field as your numbers recovers the theory of proportionality. This does *not* rely on the Archimedean axiom.

But of course the pseudo-field could be wildly different from the real numbers.

Let me spell this out a bit more. A crucial distinction between solutions 1) and 2) is the role of number. In the metric geometry approach ratio's are numbers. In Euclid's approach one only has the 4-ary relation $AB : CD :: EF : GH$; $AB : CD$ or AB/CD does not mean anything to Euclid. Hilbert resolves this by starting like Euclid in a completely geometric setting. Then he defines the set of numbers as the set of equivalence classes under congruence of segments. He proves that with the natural notions of addition of multiplication this set of 'numbers' is field. So he can think of the 'length' of the segment AB as the equivalence class of all segments congruent to AB . It is easy then to see that corresponding sides of similar triangles are proportional, that is the dividing their lengths (in the field we just introduced) gives the same answer. (Because that is how we defined multiplication.)

This is an extremely elegant solution. Equally clearly defining fields and equivalence classes etc. is not a subject for 10th grade. My point is that this show the writer of a geometry text that the theory of proportion can be established without knowing anything about the theory of limits. Thus it is not essential to introduce the real numbers (as in the metric approach) in order to have a reasonable treatment of proportionality.

So any treatment of proportionality in high school is going to at best touch on the notion of incommensurability.

3 Constructions

Constructions are basic to Euclid; they are an adjunct to the metric approach. This seems to be because neither the metric approach nor Hilbert really justifies the construction of a circle with an arbitrary center and radius.

Euclid assumes that he can construct a circle with given radius and proves in Propositions I and II, that he can construct on any line through any point a segment congruent to a given segment.

Hilbert and the metric geometry school both assume you can construct segments and angles of arbitrary size at an arbitrary point. (This is implicit in [7] and explicit in books with the protractor or ruler postulate.)

Hilbert does however justify the construction of a perpendicular bisector and thus of a circle through three given points. We sketch this argument below. The existence of a perpendicular is a nice result in Hilbert's system; See [6].

1) Prove the exterior angle theorem: an exterior angle to a triangle is greater than either of the opposite interior angles. We gave Hilbert's nice construction in class.

2) Conclude that in a triangle ABC if AB is shorter than AC then $\angle ACB < \angle CBA$.

3) With Fact 2) we can construct an isosceles triangle on any segment AB . (Draw any triangle ABC ; if it is not isosceles construct a smaller triangle with equal angles and use ASA.)

Remark 8 *The next result is easy if we assume the parallel postulate in the form that the sum of the angles of a triangle is 180 degrees. Surprisingly (to me), the parallel postulate is not needed.)*

4) AAS (If two corresponding angle and one pair of sides are congruent the triangle is congruent.) Proof. Suppose $AB \cong DE, \angle BAC \cong EDF, \angle ACB \cong DFE$. Suppose, there is an F' on AC with $AE' \cong DF$. (If not, we reverse the roles of the two triangles). Draw $F'B$. By SAS, By SAS $ABF' \cong DEF$. Thus $\angle AF'B \cong DFE$. But by the exterior angle theorem $\angle AF'B > \angle F'CB = \angle ACB$. This contradicts $\angle ACB \cong \angle DFE$.

5) Construct the midpoint of a given segment AB . (Draw any point D on one side of AB and a line BD . On the other side of AB choose E so that $AE \cong BD$. Draw ED ; the intersection with AB is the midpoint. (Use 4) to justify the last claim.)

6) To construct the perpendicular bisector of AB , construct an isosceles triangle ABC and a midpoint M of AB . The line through BM is the perpendicular bisector.

3.1 Independence

The key idea is to understand what it means for a proposition to be independent from another set of propositions [1, 5]. We gave several examples.

In a non-Archimedean plane, two triangles on the same base and with the same height can have different areas [5].

Interpreting the geometric axioms on the surface of a sphere,

1. Two distinct lines can intersect in two points.
2. The parallel postulate fails, there are no parallel lines
3. There are two triangles with the same base and same height but different areas.

4 Pedagogy

The main claim of the course is that the choices of axiom system seriously impact what students can learn in the geometry class.

If the real numbers are built into the system there is a temptation to insist on very pedantic proofs of trivial facts.

If SSS and ASA are axioms, two accessible and interesting theorems have been just assumed.

A geometry that is not metric in the sense above (e.g. Euclid or Hilbert) has the disadvantage that one must talk (at least implicitly) about equivalence classes of congruent segments rather than having a fixed length for a segment. It has the advantage of showing that the numbers are not essential.

The strange examples we have given of the meanings of phrases like the height of a triangle should emphasize the importance of finding your students mental image of fundamental geometric terms and making sure it conforms with the definition in your classroom.

What are three meanings of parallel? Does the notion of height of a triangle make sense if you have not specified a base.

Most of the following references are to items on the webpage for the course. The ones without authors are my lecture notes on webpage.

References

- [1] on webpage *Formalization of axioms*
- [2] David Hilbert. *Foundations of geometry*. Open Court Publishers, 1971. German publication 1899. Also on web
- [3] Euclid on webpage *Euclid's Elements*
- [4] on webpage Coordinatization of ordered fields
- [5] on webpage Independence
- [6] Robin Hartshorne. *Geometry: Euclid and Beyond*. Springer-Verlag, 2000.
- [7] Rhoad, Milkaukas, and Whipple. *Geometry for enjoyment and challenge*. McDougal-Littell & Company, 1901.
- [8] on webpage Superposition
- [9] on webpage Proof Systems and Completeness
- [10] on webpage Parallels and Similarity