The theory of proportionality

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Break into groups (Jose, Kristin, Elaina, Carlo, Alex) (Dana, Sheila, Jon, Sarah) (Rachel,Dhvani, Zorah, Peter, Ron)

I write below a summary and some questions concerning much of what we have done so far this semester. The groups should discuss this document and develop questions about it. Identify any mathematical statements that still must be proved in class to fill out the argument. What historical or pedagodical claims need support? Does the overall argument make sense? What are specific points that you find troubling? I would like written questions or comments from each group.

Theories of Proportionality

Algebra was integrated into geometry in the 60's. This led to difficulties in understanding proof. Geometry books went through a phase of more and more confusion in the actual axiom systems and development of geometry. Then, a trend towards eliminating proof set in. We seek to understand why algebra was integrated with geometry.

Specifically, 'by integrating algebra into geometry', I mean including the axioms for the real arithmetic via the ruler postulate. Part of this is to deal with metric geometry; segments have definite lengths.

That is, in the 'modern' (Birkhoff-Moise) treatment, the length of a line segment is a basic notion. This contrasts with Euclid and Hilbert where congruence is basic and length is only implicit.

Raimi says such an integration was necessary because the theory of limits were used to define proportions of incommensurables and limits were treated badly in early 20th century books. The implication is that a theory of limits is necessary to have a coherent theory of proportion that involves incommensurables. One might infer that limits are necessary to treat proportions of incommensurables. This is false on two grounds:

a) Euclid/Eudoxus show how to treat this case; there are two difficulties. The notation is appalling because there is no object representing a ratio (we don't have reals.) And because they seem to rely on the axiom of Archimedes.

b) Hilbert shows that by coordinatizing the plane with an (almost– no subtraction) field one gets a well-grounded theory of proportion. (Hilbert can be modified to give a field.)

To make sense of the previous claim, we need to know what we mean by a 'coherent theory of proportion'. I propose that we mean a set of axioms and a definition of 'proportional' in which we can deduce the following propositions.

1. triangles on the same base with the same height have the same areas;
2. corresponding sides of similar triangles are proportional;
3. for triangles of the same heights, the areas are proportional to the bases.

In item 1) what does it mean for the triangle to have the same area?

How does Hilbert's procedure of defining addition and multiplication fit into our earlier discussion of models of a first order sentence?

We will deal later with why Hilbert considers the axioms of Archimedes and 'completeness'? Moise notes that under the Archimedean axiom this field is a subfield of the reals (and if the geometry also satisfies a suitable geometric translation of Dedekind completeness) it is the reals.