MCS 261, Dec. 13, 2002 Name: ____________________

Final Exam

- ○ There are ten (10) problems on this exam.
- ○ The number of points each problem is worth is listed by the problem.
- ○ Write your solutions in your exam book.
- ○ Turn in this sheet along with your solutions.
- ○ Show and explain all of your work.
- ○ **An unjustified answer is not correct!**

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(1) (10 points) Let $A = \{1, 2, 3, 5\}$ and $B = \{2, 4\}$. Find $A \cup B$, $A \cap B$, $A \backslash B$. List two elements of $A \times B$. What is the cardinality of $A \times B$.

Solution: $A \cup B = \{1, 2, 3, 4, 5\}$, $A \cap B = \{2\}$, $A \backslash B = \{1, 3, 5\}$. Two elements of $A \times B$ are $\langle 2, 2\rangle$ and $\langle 5, 2\rangle$. $A \times B$ has 8 elements.

(2) (10 points) Recall Z is the set of integers. Let $\mathcal{R}(x, y)$ hold if the remainder when x is divided by 11 is the same as the remainder when y is divided by 11. Is $\mathcal{R}$ an equivalence relation on Z? Why or why not?

Solution: Let a % 11 denote the remainder when a is divided by 11. Then a % 11 = a % 11 so R is reflexive. Moreover if a % 11 = b % 11 then b % 11 = a % 11 so R is symmetric. Finally if a % 11 = b % 11 and b % 11 = c % 11 then a % 11 = c % 11 so R is transitive. Thus, R is an equivalence relation,.

(3) Just write the answers to this question in terms of binomial coefficients. Explain why you used a certain formula. Don't waste time multiplying it out.

(a) (8 points) Bob is going to choose a selection of 12 chocolates. There are 25 varieties and he can have as many as he wants of each kind. How many ways can he make the selection?

Solution: We are distributing 12 identical marbles in 25 holes with an arbitrary number of marbles in each hole. The number of ways to do this is $\binom{n+r-1}{r} = \binom{36}{12}$.

(b) (12 points) A committee of 7 is being picked from Congress, At most one of the 7 members will be one of the 52 members of the California delegation. The others will from the remaining 374 members of congress. How many ways are there to pick the committee. Explain your reasoning.

Solution: There are $\binom{374}{7}$ ways to choose a committee with no member from California and $52 \cdot \binom{374}{6}$ ways to choose a committee with one member from California. These possibilities are disjoint so the answer is: $\binom{374}{7} + 52 \cdot \binom{374}{6}$.

(4) (20 points) Let S be the set of nonnegative natural numbers which are perfect squares. List some members of S. Prove that S has the same cardinality as the set N of all natural numbers $N = 0, 1, 2, \ldots$.

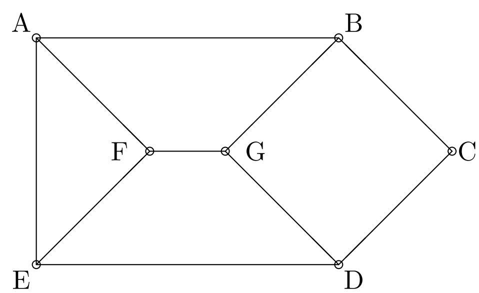
Solution: Let f map N to S be defined by $f(x) = x^2$. Some typical members of S are $4, 25, 625$. Since each member of S has a unique *positive* square root this function is 1-1. Each member of S is a perfect square so it is in the range of f and so f is onto. Since we have a 1-1, onto function from N to S they have the same cardinality.

(5) (20 points) Sixteen points are chosen inside a 5×3 rectangle. Prove that two of these points lie within $\sqrt{2}$ of each other.

Solution: The area of the rectangle is 15 square units. By the pigeonhole principle, two of the points must lie in the same 1 by 1 square; therefore they are no further apart than the diagonal of the square: $\sqrt{2}$.

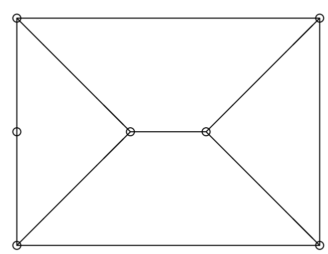
(6) (20 points) Here is a graph G_1.

G_1

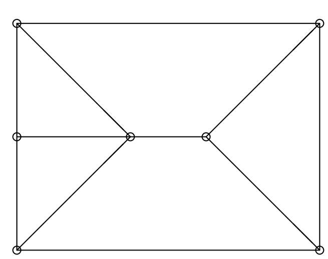


For each of the following graphs G_2, G_3 either label it to show it is isomorphic to G_1 or explain why no isomorphism exists. Note that in both G_1 and G_2 there are three vertices on the left hand side of the graph.

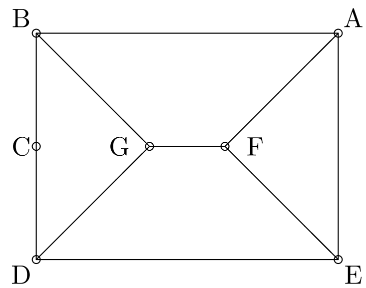
G_2



G_3



Solution: G_1 and G_3 are not isomorphic because G_3 has more edges. G_1 and G_2 are seen to be isomorphic by the following labeling.



(7) (15 points) Let p and q be statements. Is the following statement a tautology? Justify your answer.

$$[p \wedge (p \rightarrow q)] \rightarrow q$$

Solution: It is a tautology. The most routine and easiest argument is to compute a truthtable. The most inventive correct solution (proferred to this exam) is to note that this is the translation of the valid rule of inference *modus ponens* into a propostional formula.

(8) (20 points) Two integers differ by 2. Multiply them and add 1 to the product. Prove the result is a perfect square.

Solution: Let x be one number and $x+2$ the other. Then their product plus one is $(x)(x+2)+1 = x^2+2x+1$ which factors as $(x+1)^2$ and so is a perfect square.

(9) (20 points) Prove $\Sigma_{k=0}^{k=n}\binom{n}{k} = 2^n$. (Hint: This can be done either considering the set of all subsets of n or by a clever use of the binomial theorem.)

Solution. binomial theorem method: $2^n = (1+1)^n = \Sigma_{k=0}^{k=n}\binom{n}{k}1^k1^{n-k} = \Sigma_{k=0}^{k=n}\binom{n}{k}$.

counting sets method: There are 2^n subsets of n. For each $k \leq n$ there are $\binom{n}{k}$ subsets of n with size k. Adding up we have: $\Sigma_{k=0}^{k=n}\binom{n}{k} = 2^n$.

Attempts to prove this by induction are quite difficult because one must consider $\Sigma_{k=0}^{k=n}\binom{n}{k}$ and $\Sigma_{k=0}^{k=n+1}\binom{n+1}{k}$.

(10) (20 points) A sequence is defined by the recursion relation

$$\begin{aligned} a_0 &= 3 \\ a_{n+1} &= 2a_n + 1 \text{ for } n \geq 0. \end{aligned}$$

Use mathematical induction to prove that

$$a_n = 2^{n+2} - 1$$

for all $n \geq 0$.

Solution. Note that $2^2 - 1 = 3$ so the propostition is true for $n = 0$. Suppose it is true for $n = k$,$a_k = 2^{k+2} - 1$. We compute a_{k+1}. By the recursion equation $a_{k+1} = 2a_k + 1$. By the induction hypothesis $2a_k + 1 = 2(2^{k+2} - 1) + 1 = 2^{k+3} - 2 + 1 = 2^{k+3} - 1$ as required. Since truth of the condition for k implies truth for $k+1$ and it true for $k = 0$, by mathematical induction it is true for all natural numbers.