FROM 5.2:

8. (a) $a_2 = 1(1+1) = 2$, $a_3 = 2(2+1) = 6$, $a_4 = 3(6+2) = 24$, $a_5 = 4(24+6) = 120$.

   (b) Guess $a_n = n!$. *Base Cases.* $n = 0$ and $n = 1$. $a_0 = 1$ by definition and $0! = 1$ by convention. $a_1 = 1$ by definition and $1! = 1$. We get equality in both cases so it is true for $n = 0$ and $n = 1$. *inductive Step.* Assume that $a_k = k!$ for *all* $0 \leq k \leq n$ and prove that $a_{n+1} = (n+1)!$. In particular, we will assume $a_n = n!$ and $a_{n-1} = (n-1)!$. By definition, $a_{n+1} = n(a_n + a_{n-1})$. By the induction hypothesis, this is equal to $n(n!+(n-1)!) = n(n\cdot(n-1)!+(n-1)!) = n(n+1)(n-1)! = (n+1)!$. Thus by strong induction, it is true for all $n \geq 0$.

35. The goal here is to prove that $\Phi^{n+1} = \Phi^n + \Phi^{n-1}$.

$$\begin{aligned}
\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} &= \left(\frac{1+\sqrt{5}}{2}+1\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \\
&= \left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^2\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}
\end{aligned}$$

FROM 5.3:

11. (a) The characteristic polynomial is $x^2 + 8x + 16 = (x+4)^2$ which has a double root of -4. So the general form of the solution is $a_n = c_1(-4)^n + c_2 n(-4)^n$. We use the initial conditions to find c_1 and c_2. $a_0 = 5 = c_1(-4)^0 + c_2(0)(-4)^0 = c_1$. $a_1 = 17 = c_1(-4)^1 + c_2(1)(-4)^1 = 5(-4) - 4c_2$ so $c_2 = -\frac{37}{4}$. The solution is

$$a_n = 5(-4)^n - \frac{37}{4}n(-4)^n = 5(-4)^n + 37n(-4)^{n-1}.$$

16. (b) To find a particular solution, try $p_n = (a+bn)2^n$. We must have $p_n = 4p_{n-1} + 3n2^n$, that is, $(a+bn)2^n = 4(a+b(n-1))2^{n-1} + 3n2^n$. Divide by 2^n to get $a + bn = 2a + 2bn - 2b + 3n$. Collect coefficients of n to get $n(b+3) + (a-2b) = 0$. We must have $b + 3 = 0$ so $b = -3$. We must have $a - 2b = 0$ so $a = 2b = -6$. The particular solution is $p_n = -(6+3n)2^n$. We now need a solution of the homogeneous recurrence relation $a_n = 4a_{n-1}$. The characteristic polynomial is $x^2 - 4x = x(x-4)$ which has roots 0 and 4. The homogeneous solution is $q_n = c_1 4^n + c_2 0^n = c_1 4^n$. So the general solution is

$a_n = p_n + q_n = -(6+3n)2^n + c_1 4^n$. We use the initial condition to solve for c_1. $a_0 = 4 = -(6+0)2^0 + c_1 4^0 = -6 + c_1$ so $c_1 = 10$. The solution is

$$a_n = 10 \cdot 4^n - (6+3)2^n.$$

20. (a) There are 3 moves required for the $n = 2$ case and 7 required for the $n = 3$ case. The tables for each case are given below.

$n = 2$	A	B	C
Initial position	1,2	*	*
Move 1	2	1	*
Move 2	*	1	2
Move 3	*	*	1,2

$n = 3$	A	B	C
Initial position	1,2,3	*	*
Move 1	2,3	*	1
Move 2	3	2	1
Move 3	3	1,2	*
Move 4	*	1,2	3
Move 5	1	2	3
Move 6	1	*	2,3
Move 7	*	*	1,2,3

(b) To move n disks from peg A to peg C, we first move the smallest $n-1$ disks to peg B, which takes a_{n-1} moves. We move the largest disk to peg C, which takes one move. We move the smallest $n-1$ disks to peg C, on top of the largest disk, which takes another a_{n-1} moves. Thus we have that $a_n = 2a_{n-1} + 1$. The initial condition is that it takes one move to move 1 disk, so $a_1 = 1$.

(c) For a particular solution, try $p_n = c$. We need $p_n = 2p_{n-1} + 1$ so $c = 2c + 1$ so $c = -1$. The particular solution is $p_n = -1$. We now need a solution of the homogeneous recursion relation $a_n = 2a_{n-1}$. The characteristic polynomial is $x^2 - 2x = x(x-2)$ which has roots 2 and 0. The solution is $q_n = c_1 2^n + c_2 0^n = c_1 2^n$. The general solution is $a_n = p_n + q_n = -1 + c_1 2^n$. We use the initial condition to find c_1. $a_1 = 1 = -1 + c_1 2^1$ so $2c_1 = 2$ and $c_1 = 1$. The solution is

$$a_n = 2^n - 1.$$

(d) To move 8 disks, it will take $2^8 - 1 = 255$ seconds, or 4.25 minutes. To move 16 disks, it will take $2^{16} - 1 = 65536$ seconds, or 1092.25 minutes, or about 18.2 hours. To move 32 disks, it will take $2^{32} - 1$ seconds, which is about 136.2 years. To move 64 disks, it will take $2^{64} - 1$ seconds, which is about 5.8×10^{11} years, that is, over 580,000 million years.

FROM 6.1:

5. Let P, G, and C be the sets of children who purchased popsicles, gum, and candy bars, respectively. Then we should have that the number of students who purchased at least one item was

$$\begin{aligned} |P \cup G \cup C| &= |P| + |G| + |C| - |P \cap G| - |P \cap C| - |G \cap C| + |P \cap G \cap C| \\ &\leq |P| + |G| + |C| - |P \cap G| - |P \cap C| - |G \cap C| \\ &= 10 + 7 + 12 - 5 - 6 - 2 = 16. \end{aligned}$$

So the clerk definitely made a mistake since there were only 15 children.

10. We are given that
   - $|U| = 75$.
   - $|A_1| = |A_2| = |A_3| = |A_4| = 28$.
   - $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4| = 12$.
   - $|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = 5$.
   - $|A_1 \cap A_2 \cap A_3 \cap A_4| = 1$.

(a) We want $|(A_1 \cup A_2 \cup A_3 \cup A_4)^c| = |U| - \sum_{1 \leq i \leq 4} |A_i| + \sum_{1 \leq i < j \leq 4} |A_i \cap A_j| - \sum_{1 \leq i < j < k \leq 4} |A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4| = 75 - 4 \cdot 28 + 6 \cdot 12 - 4 \cdot 5 + 1 = 16$.

(b) We want the number of elements that are in exactly two sets. Since all of the sets and their intersections have the same size, we can find the number of elements that are in A_1 and A_2 but not in A_3 and A_4 and multiply this number by 6 (there are 6 pairs of sets). This number is

$$\begin{aligned} & |(A_1 \cap A_2) \backslash (A_3 \cup A_4)| \\ = & |A_1 \cap A_2| - |(A_1 \cap A_2) \cap (A_3 \cup A_4)| \\ = & |A_1 \cap A_2| - |((A_1 \cap A_2) \cap A_3) \cup ((A_1 \cap A_2) \cap A_4)| \\ = & |A_1 \cap A_2| - (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| - |(A_1 \cap A_2 \cap A_3) \cap (A_1 \cap A_2 \cap A_4)|) \\ = & |A_1 \cap A_2| - |A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_2 \cap A_3 \cap A_4| \\ = & 12 - 5 - 5 + 1 = 3 \end{aligned}$$

So the number of elements in any given pair of subsets, but not the others, is 3. The total number of elements that are in exactly two subsets is $6 \cdot 3 = 18$.

15. Let $A = \{n | 0 \leq n \leq 10,000, 3|n\}$, $B = \{n | 0 \leq n \leq 10,000, 5|n\}$, $C = \{n | 0 \leq n \leq 10,000, 7|n\}$, and $D = \{n | 0 \leq n \leq 10,000, 11|n\}$. First find the size of

each set and all the intersections.

$$\begin{aligned}
|A| &= \lfloor\frac{10,000}{3}\rfloor = 3333\\
|B| &= \lfloor\frac{10,000}{5}\rfloor = 2000\\
|C| &= \lfloor\frac{10,000}{7}\rfloor = 1428\\
|D| &= \lfloor\frac{10,000}{11}\rfloor = 909\\
|A\cap B| &= \lfloor\frac{10,000}{3\cdot 5}\rfloor = 666\\
|A\cap C| &= \lfloor\frac{10,000}{3\cdot 7}\rfloor = 476\\
|A\cap D| &= \lfloor\frac{10,000}{3\cdot 11}\rfloor = 303\\
|B\cap C| &= \lfloor\frac{10,000}{5\cdot 7}\rfloor = 285\\
|B\cap D| &= \lfloor\frac{10,000}{5\cdot 11}\rfloor = 181\\
|C\cap D| &= \lfloor\frac{10,000}{7\cdot 11}\rfloor = 129\\
|A\cap B\cap C| &= \lfloor\frac{10,000}{3\cdot 5\cdot 7}\rfloor = 95\\
|A\cap B\cap D| &= \lfloor\frac{10,000}{3\cdot 5\cdot 11}\rfloor = 60\\
|A\cap C\cap D| &= \lfloor\frac{10,000}{3\cdot 7\cdot 11}\rfloor = 43\\
|B\cap C\cap D| &= \lfloor\frac{10,000}{5\cdot 7\cdot 11}\rfloor = 25\\
|A\cap B\cap C\cap D| &= \lfloor\frac{10,000}{3\cdot 5\cdot 7\cdot 11}\rfloor = 8
\end{aligned}$$

(a) We want $|A\cup B\cup C\cup D| = |A|+|B|+|C|+|D|-|A\cap B|-|A\cap C|-|A\cap D|-|B\cap C|-|B\cap D|-|C\cap D|+|A\cap B\cap C|+|A\cap B\cap D|+|A\cap C\cap D|+|B\cap C\cap D|-|A\cap B\cap C\cap D| = 3333+2000+1428+909-666-476-303-285-181-129+95+60+43+25-8 = 5845$.

(b) Here we want $|(A\cap B)\backslash(C\cup D)| = |A\cap B|-|A\cap B\cap C|-|A\cap B\cap D|+|A\cap B\cap C\cap D| = 666-95-60+8 = 519$. See problem 10 for the derivation of this expression.

(c) For each interesection of three sets, we find the number of elements in that intersection that are not also in the fourth set.

$$\begin{aligned}
|(A\cap B\cap C)\backslash D| &= |A\cap B\cap C|-|A\cap B\cap C\cap D| = 95-8 = 87\\
|(A\cap B\cap D)\backslash C| &= |A\cap B\cap D|-|A\cap B\cap C\cap D| = 60-8 = 52\\
|(A\cap C\cap D)\backslash B| &= |A\cap C\cap D|-|A\cap B\cap C\cap D| = 43-8 = 35\\
|(B\cap C\cap D)\backslash A| &= |B\cap C\cap D|-|A\cap B\cap C\cap D| = 25-8 = 17
\end{aligned}$$

These are the four possible combinations of three sets that an element could belong to. We add them up for the answer: $87 + 52 + 35 + 17 = 191$.

(d) The elements that are in at most three sets cannot be in all four, so we subtract $|A \cap B \cap C \cap D|$ from the size of the universe: $10,000 - 8 = 9992$.