From 3.1:

11. (a) h is not one-to-one since $h(1) = h(-1) = 3$, but $1 \neq -1$, and both 1 and -1 are in the domain. h is not onto since 1 is in the target, but there is no integer x such that $h(x) = x^2 + 2 = 1$.

    (b) In this case, h is one-to-one since if $h(x_1) = h(x_2)$, then $x_1^2 + 2 = x_2^2 + 2$ so $x_1^2 = x_2^2$ and $x_1 = \pm x_2$. Since x_1 and x_2 are natural numbers, they must be equal. h is still not onto because 1 is in the target but not the range.

16. (a) From the graph is is clear that g is onto but not one-to-one.

    (b) g is not one-to-one since $g(-1) = g(1) = 1$ but $-1 \neq 1$ and 1 and -1 are both in the domain. g is also not onto because 2 is in the target but not the range: suppose there is an integer x such that $g(x) = 2$. Then $x^3 - x + 1 = 2$ so $x(x^2 - 1) = 1$. For the product of two integers to be 1, either both must be 1 or both -1. Both of these cases are impossible.

    (c) By the same argument as in (b), g is not onto. g is, however, one-to-one. Suppose $g(x_1) = g(x_2)$. Then $x_1^3 - x_1 + 1 = x_2^3 - x_2 + 1$ so $x_1^3 - x_2^3 = x_1 - x_1$. If $x_1 \neq x_2$, then we can divide both sides by $x_1 - x_2$ to get $x_1^2 + x_1 x_2 + x_2^2 = 1$. Since x_1 and x_2 are natural numbers, this is impossible – we would have to have $x_1^2 + x_1 x_2 + x_2^2 \geq 3$. Thus we must have $x_1 = x_2$ so g is one-to-one.

21. (a) $\sim$ is reflexive because $f(a) = f(a)$ for all $a \in A$. $\sim$ is symmetric because if $a \sim b$, then $f(a) = f(b)$ so $f(b) = f(a)$ and $b \sim a$. $\sim$ is transitive because if $a \sim b$ and $b \sim c$, then $f(a) = f(b) = f(c)$ so $a \sim c$. Thus $\sim$ is reflexive, symmetric, and transitive, so it is an equivalence relation.

    (b) $f(0) = \lfloor 0 \rfloor = 0$ so $\overline{0}$ is the set of all real numbers with $\lfloor x \rfloor = 0$, that is, the interval $[0, 1)$. $f(\frac{7}{5}) = \lfloor \frac{7}{5} \rfloor = 1$ so $\overline{\frac{7}{5}}$ is the set of all real numbers with $\lfloor x \rfloor = 1$, that is, the interval $[1, 2)$. $f(-\frac{3}{4}) = \lfloor -\frac{3}{4} \rfloor = -2$ so $\overline{-\frac{3}{4}}$ is the set of all real numbers with $\lfloor x \rfloor = -2$, that is, the interval $[-2, -1)$.

    (c) The conjugacy classes are the sets of all elements that have the same image. They are $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$.

From 3.2:

4. (a) f is onto because if $y = 2k$ is an even natural number, there is an $x = -k$ such that $f(x) = y$, and if $y = 2k + 1$ is an odd natural number, there is an $x = k$ such that $f(x) = y$. f is one-to-one for the following reason. Suppose $f(x_1) = f(x_2)$. This number is either even or odd. If it is even, then $f(x_1) = f(x_2) = 2x_1 + 1 = 2x_2 + 1$ so $x_1 = x_2$. If it is odd, then $f(x_1) = f(x_2) = 2|x_1| = 2|x_2|$ and x_1 and x_2 are both negative, so $x_1 = x_2$. The function is one-to-one and onto so it has an inverse.

    (b) $f^{-1}(2586)$ is the x such that $f(x) = 2586$. Since this is even, we have $2586 = 2|x|$ with $x < 0$ so $x = 1293 = f^{-1}(2586)$.

6. (a) $f \circ g = \{(1,1),(2,1),(3,8),(4,9),(5,1)\}$. $g \circ f$ is not defined because the range of f is not the domain of g. $f \circ f$ is not defined because the domain and range of f are different. $g \circ g = \{(1,2),(2,2),(3,2),(4,1),(5,2)\}$.

   (b) f is one-to-one because different elements of S have different images, and it is onto because every element of T is the image of some element of S. g is not one-to-one because 2 is the image of more than one element of S, and it is not one-to-one because 4 is in S but not in the range of g.

   (c) $f^{-1} = \{(1,2),(2,5),(3,4),(8,1),(9,3)\}$.

   (d) g^{-1} does not exist because g is neither one-to-one or onto.

9. $g \circ f(x) = g(f(x)) = g\left(\frac{x}{x+1}\right) = \frac{1}{\frac{x}{x+1}} = \frac{x+1}{x}$.
   $f \circ g(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}}{\frac{1}{x}+1} = \frac{1}{x+1}$.
   $f \circ g \circ h(x) = (f \circ g)(h(x)) = (f \circ g)(x+1) = \frac{1}{(x+1)+1} = \frac{1}{x+2}$.

FROM 3.3:

12. (a) Define $f : (0,1) \to (1,2)$ by $f(x) = x+1$. We show this is a one-to-one correspondence. Suppose $f(x_1) = f(x_2)$. Then $x_1 + 1 = x_2 + 1$ so $x_1 = x_2$ and f is one-to-one. Given any $y \in (1,2)$, there is an $x = y - 1$ in $(1,0)$ such that $f(x) = y$, so f is onto. Therefore there is a one-to-one correspondence between the two intervals so they have the same cardinality.

    (b) Define $f : (0,1) \to (4,6)$ by $f(x) = 2x+4$. We show this is a one-to-one correspondence. Suppose $f(x_1) = f(x_2)$. Then $2x_1 + 4 = 2x_2 + 4$ so $x_1 = x_2$ and f is one-to-one. Given any $y \in (4,6)$, there is an $x = \frac{1}{2}y - 2$ in $(0,1)$ such that $f(x) = y$, so f is onto. Therefore there is a one-to-one correspondence between the two intervals so they have the same cardinality.

    (c) Define $f : (a,b) \to (c,d)$ by $f(x) = \left(\frac{d-c}{b-a}\right)(x-a)+c$. We show this is a one-to-one correspondence. Suppose $f(x_1) = f(x_2)$. Then $\left(\frac{d-c}{b-a}\right)(x_1 - a) + c = \left(\frac{d-c}{b-a}\right)(x_2 - a) + c$ so $x_1 = x_2$ and f is one-to-one. Given any $y \in (c,d)$, there is an $x = \left(\frac{b-a}{d-c}\right)(y-c) + a$ in (a,b) such that $f(x) = y$, so f is onto. (Note: You should check that this x *is* in (a,b)). Therefore there is a one-to-one correspondence between the two intervals so they have the same cardinality.

17. (a) $2^1, 2^{-1}, 2^2, 2^{-2}, 2^3, 2^{-3}, 2^4, 2^{-4}, 2^5, 2^{-5}, 2^6, 2^{-6}, \ldots$

18. (b) This is a subset of the rational numbers, which are countably infinitite, so this set is countable. This set contains the infinite sequence $1+\frac{1}{2}, 1+\frac{1}{3}, 1+\frac{1}{4}, \ldots$ so it is indeed infinite. Therefore it is countably infinite.

    (c) There are a finite number of possible values of m (99), and a finite number of possible values for n (100), so there are a finite number of elements in this set. In fact, there are at most than 9900 of them.