From 4.2:

18. We will use the Euclidean algorithm.

$$\begin{aligned} 5k+3 &= 1\cdot(3k+2)+(2k+1)\\ 3k+2 &= 1\cdot(2k+1)+(k+1)\\ 2k+1 &= 1\cdot(k+1)+k\\ k+1 &= 1\cdot k+1\\ k &= k\cdot 1+0 \end{aligned}$$

The last nonzero remainder is 1 so it seems that $\gcd(5k+3, 3k+2) = 1$. We just need to check that we used the division algorithm correctly at each step. But this is cleared up by noting that for $k > 0$, $0 \le 2k+1 < 3k+2$, $0 \le k+1 < 2k+1$, and $0 \le k < k+1$. in addition, if $k > 1$, $0 \le 1 < k$.

22. Assume that $\gcd(a, c) = 1$, and $b|c$. Let's call $\gcd(a, b) = g$. If g is the greatest common divisor of a and b, the $g|a$ and $g|b$. By the transitivity of divisibility, we have that $g|b$ and $b|c$ so $g|c$. So $g|a$ and $g|c$, so $g \le gcd(a, c) = 1$ so $gcd(a, b) = 1$.

From 4.4:

3. (b) $43,197 = 129\cdot 333 + 240$ so $43197 \equiv 240 \pmod{333}$.
(d) $-125,617 = -399\cdot 315 + 68$ so $-125,617 \equiv 68 \pmod{315}$.
(e) $11,111,111,111 = 10,001,000\cdot 1111 + 111$ so $11,111,111,111 \equiv 111 \pmod{1111}$.

6. (c) $17,123 = 2853\cdot 6 + 5$ so $17,123 \equiv 5 \pmod 6$. Using this and the fact that $5^5 = 3125 \equiv 5 \pmod 6$, we have

$$(17,123)^{50} \equiv 5^{50} \equiv \left(5^5\right)^{10} \equiv 5^{10} \equiv \left(5^5\right)^2 \equiv 5^2 \equiv 1 \pmod 6.$$

9. (b) We can't divide by 2 since $\gcd(2, 6) \ne 1$, and multiplying doesn't give anything nicer either, so let's make a table:

x	0	1	2	3	4	5
$4x \equiv$	0	4	2	0	4	2

So $x \equiv 2 \pmod 6$ and $x \equiv 5 \pmod 6$ are the solutions.
(d) We notice from the table in (a) that there are no values of x such that $4x \equiv 3 \pmod 6$, so there is no solution.

11. (d) Multiply the first congruence by 2 to get $14x+4y \equiv 6 \pmod{15}$. Subtract the second equivalence from this one to get $5x \equiv 0 \pmod{15}$. This has solutions $x \equiv 0, 3, 6, 9, 12 \pmod{15}$.

    If $x \equiv 0 \pmod{15}$, we have $2y \equiv 3$ and $4y \equiv 6 \pmod{15}$. The second is just twice the first, so we solve the first. Notice that $2y \equiv 3 \equiv 18 \pmod{15}$. Since $\gcd(2, 15) = 1$, we can divide by 2 to get $y \equiv 9 \pmod{15}$. One solution is $x \equiv 0$ and $y \equiv 9$.

    If $x \equiv 3 \pmod{15}$, we have $2y \equiv 3 - 21 \equiv 12$ and $4y \equiv 6 - 27 \equiv 9 \pmod{15}$. The second is twice the first so solve the first by dividing by 2: $y \equiv 6 \pmod{15}$. One solution is $x \equiv 3$ and $y \equiv 6$.

    If $x \equiv 6 \pmod{15}$, we have $2y \equiv 3 - 42 \equiv 6$ and $4y \equiv 6 - 54 \equiv 12 \pmod{15}$. The second is twice the first so solve the first by dividing by 2: $y \equiv 3 \pmod{15}$. One solution is $x \equiv 6$ and $y \equiv 3$.

    If $x \equiv 9 \pmod{15}$, we have $2y \equiv 3 - 63 \equiv 0$ and $4y \equiv 6 - 81 \equiv 0 \pmod{15}$. The second is twice the first so solve the first by dividing by 2: $y \equiv 0 \pmod{15}$. One solution is $x \equiv 9$ and $y \equiv 0$.

    If $x \equiv 12 \pmod{15}$, we have $2y \equiv 3 - 84 \equiv 9$ and $4y \equiv 6 - 108 \equiv 3 \pmod{15}$. The second is twice the first so we solve the first by dividing the following by 2: $2y \equiv 9 \equiv 24 \pmod{15}$ so $y \equiv 12 \pmod{15}$. One solution is $x \equiv 12$ and $y \equiv 12$.

    The above cover all possible cases so the solutions are: $x \equiv 0, y \equiv 9$; $x \equiv 3, y \equiv 6$; $x \equiv 6, y \equiv 3$; $x \equiv 9, y \equiv 0$; $x \equiv 12, y \equiv 12$.

15. (b) The statement is true. We assume that $a \equiv b \pmod{n}$ and want to prove that $a^2 \equiv b^2 \pmod{n}$. If $a \equiv b \pmod{n}$, then $n|(a-b)$ so $a - b = nk$ for some integer k. $a^2 - b^2 = (a-b)(a+b) = nk(a+b) = n[k(a+b)]$. $k(a+b)$ is an integer so $n|(a^2 - b^2)$ thus $a^2 \equiv b^2 \pmod{n}$.

FROM 5.1:

3. *Base Case:* We can fill an order for $n = 32$ pounds of fish by using one 5-pound and 3 nine-pound fish.

    Inductive Step: We assume that we can fill an order of n fish. We need to show that we can fill an order of $n+1$ fish.

    - If our order for n fish has a 9-pound fish, replace it with two 5-pound fish. We replaced 9 pounds with 10 pounds, so the total number of pounds increases from n to $n+1$.
    - If our order for n fish has no 9-pound fish, then since $n \geq 32$, there must be at least seven 5-pound fish. Replace these with four 9-pound fish. We replaced 35 pounds with 36 pounds, so the total number of pounds increases from n to $n+1$.

    So we can fill an 32-pound order and if we can fill an n-pound order then we can fill an $n+1$-pound order, so we can fill any order of n pounds for $n \geq 32$.

4. (b) *Base Case:* For $n = 1$, $n^3 + 2n = 1^3 + 2 \cdot 1 = 1 + 2 = 3$ is divisible by 3, so the statement is true for $n = 1$.

Inductive Step: We assume that the statement is true for n, that is, we assume that $n^3 + 2n$ is divisible by 3. We prove that the statement is true for $n + 1$, that is, we prove that $(n + 1)^3 + 2(n + 1)$ is divisible by 3.

$$(n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 = (n^3 + 2n) + 3n^2 + 3n + 3.$$

We are assuming that $n^3 + 2n$ is divisible by 3, that is, that $n^3 + 2n = 3k$ for some integer k. So the above is equal to

$$(n^3 + 2n) + 3n^2 + 3n + 3 = 3k + 3n^2 + 3n + 3 = 3(k + n^2 + n + 1),$$

which is divisible by 3 since $k + n^2 + n + 1$ is an integer.

Thus the statement is true for $n = 1$ and if it is true for n then it is true for $n + 1$ so it is true for all $n \geq 1$.