Example 1 *Why Axioms III and IV are needed! big time!*

Let Π *be* $\Re \times \Re$*, the usual real plane. Let lines be the usual notion of lines. But for every line* ℓ *define:*

Ψ_ℓ^1 *is the points in the plane that are not on* ℓ *and have both coordinates rational.*

Ψ_ℓ^2 *is the points in the plane that are not on* ℓ *and have at least one irrational coordinate.*

Now Axiom II holds but the two half planes, while disjoint, are all mixed together.

Since the publishers omitted the proof of 7.46, I am posting it.

Theorem 2 (7.46) *If* $C \in {}^{o}\overline{AB}^{o}$ *then* $\overrightarrow{CB} \subseteq \overrightarrow{AB}$*.*

Proof. We show $X \notin \overrightarrow{AB}$ implies $X \notin \overrightarrow{CB}$ and apply contraposition. Without loss of generality we may assume X is on the line AB.

By 7.36, $C \in {}^{o}\overline{AB}^{o}$ implies $\overrightarrow{AC} = \overrightarrow{AB}$. Substituting, $X \notin \overrightarrow{AB}$ implies $X \notin \overrightarrow{AC}$. $X \notin \overrightarrow{AC}$ means $X \notin \Psi_m^C$ for any m with $m \cdot AB = A$. Thus $X - A - C$. So by Axiom IV, $A \in {}^{o}\overline{XC}^{o}$ and again by 7.36, $\overrightarrow{CX} = \overrightarrow{CA}$.

On the other hand, $C \in {}^{o}\overline{AB}^{o}$ implies by Axiom IV that $A - C - B$. That is,

$$\overrightarrow{CA} \cap \overrightarrow{CB} = \{C\}.$$

Since we just showed, $\overrightarrow{CX} = \overrightarrow{CA}$, substituting we have

$$\overrightarrow{CX} \cap \overrightarrow{CB} = \{C\}.$$

So $X - C - B$ and we finish.