BEYOND FIRST ORDER LOGIC I

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

June 15, 2004

ORIGINS

Zilber constructed several sentences ϕ in $L_{\omega_1,\omega}(Q)$ and

1. gave sufficient conditions for ϕ to be categorical in every uncountable cardinal;
2. these sentences (conjecturally) provide more information about complex exponentiation.

WHY GO BEYOND FIRST ORDER LOGIC?

I. Because it's there.

A. To understand the infinite:
B. To understand canonical structures

II. To understand first order logic

III. To understand 'Model Theory'

IV. To investigate ordinary mathematical structures

This is the first of two talks with different emphases on going beyond first order logic. The different foci of the talks are explained below.

TO UNDERSTAND THE INFINITE!

Most known mathematical results are either

extremely cardinal dependent: about finite or countable structures or at most structures of cardinality the continuum;

or completely cardinal independent: about every structure satisfying certain properties.

Understanding Classes of Models

Logics vrs classes of models: Robinson, Tarski, Morley, Shelah

Model theory has discovered problems that have an intimate relation between the cardinality of structures and algebraic properties of the structures:

i) Stability spectrum and counting models
ii) A general theory of independence
iii) Decomposition theorems for general models

There are structural algebraic, not merely combinatorial features, which are non-trivially cardinal dependent.

TO UNDERSTAND CANONICAL STRUCTURES

A Thesis of Zilber:

Fundamentally important structures like the complex field can be described at least up to categoricity in power in an appropriate logic.

TO UNDERSTAND FIRST ORDER LOGIC

The study of first order logic uses without thinking such methods as:

1. compactness theorem
2. upward and downward Löwenheim-Skolem theorem
3. closure under unions of Elementary Chains
4. Ehrenfeucht-Mostowski models

We can better understand these methods and their use in the first order case by investigating situations where only some of them hold.

There are important, not well understood, connections between n-dimensional amalgamation in the infinitary and first order situations.

TO UNDERSTAND MODEL THEORY

What are the *syntactial* and *semantical* components of model theory?

How does the ability to change vocabulary distinguish model theory from other mathematical disciplines?

INVESTIGATE ORDINARY MATHEMATICS

1. Banach Spaces (Krivine, Stern, Henson, Iovino et al)
2. Complex Exponentiation (Zilber)
3. Group Representations (Hytinnen-Lessmann-Shelah)
4. Locally finite groups (Grossberg, Macintyre, Shelah)

THE CATEGORICITY SPECTRUM

Theorem 1 (Morley) *A countable* first order *theory T is categorical in one uncountable cardinal if and only if it is categorical in all uncountable cardinalities.*

Is first order crucial? Does the theorem generalize to other classes of models?

Some counterexamples:

Let the vocabulary contain a unary predicate P.

In $L(Q)$ we can say both the set and its compliment are uncountable. This theory is categorical in $\aleph_1$ and nowhere else.

With an additional binary relation we can say $2^{|P(M)|} \geq |M|$. The class of reducts is categorical in κ only if $\kappa = \beth_\alpha$ for some limit ordinal α.

What kinds of classes do we mean?

PCΓ CLASSES

A class **K** of τ-structures is called PC if it is the collection of reducts to τ of the models of a first order theory T' in some $\tau' \supseteq \tau$.

A class **K** of τ-structures is called $PC\Gamma$ if it the collection of reducts to τ of the models of a first order theory T' in some $\tau' \supseteq \tau$ which omit all types in a specified collection Γ of types in finitely many variables over the empty set.

While we give the next definition semantically notice that obvious examples are the class of models of any sentence of $L_{\kappa,\omega}$, for any κ.

ABSTRACT ELEMENTARY CLASSES

Definition 2 *A class of L-structures,* $(\mathbf{K}, \preceq_{\mathbf{K}})$, *is said to be an* abstract elementary class: AEC *if both* $\mathbf{K}$ *and the binary relation* $\preceq_{\mathbf{K}}$ *are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If* $M \preceq_{\mathbf{K}} N$ *then* $M \subseteq N$.
- **A2**. $\preceq_{\mathbf{K}}$ *is a partial order on* $\mathbf{K}$.
- **A3**. *If* $\langle A_i : i < \delta \rangle$ *is* $\preceq_{\mathbf{K}}$*-increasing chain:*
  1. $\cup_{i<\delta} A_i \in \mathbf{K}$;
  2. *for each* $j < \delta$, $A_j \preceq_{\mathbf{K}} \cup_{i<\delta} A_i$
  3. *if each* $A_i \preceq_{\mathbf{K}} M \in \mathbf{K}$ *then* $\cup_{i<\delta} A_i \preceq_{\mathbf{K}} M$.
- **A4**. *If* $A, B, C \in \mathbf{K}$, $A \preceq_{\mathbf{K}} C$, $B \preceq_{\mathbf{K}} C$ *and* $A \subseteq B$ *then* $A \preceq_{\mathbf{K}} B$.
- **A5**. *There is a Löwenheim-Skolem number* $\mathrm{LS}(\mathbf{K})$ *such that if* $A \subseteq B \in K$ *there is a* $A' \in \mathbf{K}$ *with* $A \subseteq A' \preceq_{\mathbf{K}} B$ *and* $|A'| \leq \mathrm{LS}(\mathbf{K}) + |\mathrm{A}|$.

TWO DIRECTIONS

A. Strong Syntax: No Assumption of upwards Löwenheim-Skolem

Theorem 3 *[Shelah]*

1. *(For $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$) A complete $L_{\omega_1,\omega}$-sentence which has few models in $\aleph_n$ for each $n < \omega$ is excellent.*
2. *(ZFC) An excellent class has models in every cardinality.*
3. *(ZFC) Suppose that ϕ is an excellent $L_{\omega_1,\omega}$-sentence. If ϕ is categorical in one uncountable cardinal κ then it is categorical in all uncountable cardinals.*

B. AEC's with arbitrarily large models:

We focus on B in this talk.

STRUCTURAL RESULTS

It is easy to see that every AEC with Löwenheim number $\aleph_0$ is determined by its restriction to countable models. All other models can be written as direct limits of these.

But it is nontrivial to characterize those AEC in which every model that can be written as a tree of countable height of countable models. This is the **main gap**, proved in

1. first order (Shelah)
2. excellent classes (Grossberg and Hart)
3. several variants for homogeneous model theory (Grossberg, Hyttinen, Lessmann, Shelah).

CONTEXT

Conjecture: Let X be the class of cardinals in which a *reasonably defined* class is categorical.

Not both X and the complement of X are cofinal.

(Note: So, PC-classes are not ‘reasonable’.)

We know this conjecture for first order theories and for excellent classes in $L_{\omega_1,\omega}$. But it is open even for general sentences in $L_{\omega_1,\omega}$. So it is reasonable to investigate it first with quite strong hypotheses.

Of course, it is only interesting when $\mathbf{K}$ has arbitrarily large models – EM methods are applicable.

EVENTUAL CATEGORICITY

Let $H_2 = H(H(|\tau|))$.

Theorem 4 (Main Result) *If the AEC* $\mathbf{K}$ *has*

1. *ap and jep*
2. *is categorical in a successor cardinal* λ *where*
   (a) $\lambda > H_2$ *and*
   (b) *for some* $\chi < H(\tau)$ *and any* χ_1, $\mathbf{K}$ *is weakly* (χ, χ_1)*-tame*

then $\mathbf{K}$ *is categorical in every* θ *with* $H_2 \leq \theta$.

Note this result is in ZFC.

Jep is assumed for convenience.

AP is a very significant assumption

We will discuss tameness and $H(\tau)$ below.

ATTRIBUTIONS

As stated this result depends on work of Shelah [4] and Grossberg-VanDieren [3]; expositions of various key lemmas are in several of my notes [1, 2].

In particular, Shelah does not assume tameness but only proves categoricity between H_2 and λ.

The notion of inducting on cardinals is due to Grossberg-Van Dieren as well as various specific arguments attributed below. They prove upwards categoricity from two successive categoricity cardinals assuming amalgamation, arbitrarily large models and that for some χ less than the categoricity cardinals and any χ_1, $\mathbf{K}$ is weakly (χ, χ_1)-tame.

THE PRESENTATION THEOREM

Every AEC is a PCΓ

More precisely,

Theorem 5 *If K is an AEC with Lowenheim number* $\mathrm{LS}(\mathbf{K})$ *(in a vocabulary τ with $|\tau| \leq$* $\mathrm{LS}(\mathbf{K})$*), there is a vocabulary τ', a first order τ'-theory T' and a set of $2^{\mathrm{LS}(\mathbf{K})}$ τ'-types Γ such that:*

$$\mathbf{K} = \{M' \restriction L : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, if M' is an L'-substructure of N' where M', N' satisfy T' and omit Γ then $M' \restriction L \preceq_{\mathbf{K}} N' \restriction L$.

EM models

First order concepts:

Notation 6 *1. For any linearly ordered set* $X \subseteq M$ *where* M *is a* τ*-structure we write*

$$\mathrm{D}_\tau(\mathrm{X}) = \Phi$$

(diagram) for the set of τ*-types of finite sequences (in the given order) from* X*.*

2. If X *is a sequence of* τ*-indiscernibles with diagram* $\Phi = \mathrm{D}_\tau(\mathrm{X})$ *and any* τ *model of* Φ *has built in Skolem functions, then for any linear ordering* I*,*

$$EM(I, \Phi)$$

denotes the τ*-hull of a sequence of order indiscernibles realizing* Φ*.*

Morley's Omitting Types Theorem

Lemma 7 *If* $(X, <)$ *is a sufficiently long linearly ordered subset of a* τ*-structure* M*, for any* τ' *extending* τ *(the length needed for* X *depends on* $|\tau'|$*) there is a countable set* Y *of* τ'*-indiscernibles (and hence one of arbitrary order type) such that* $\mathrm{D}_\tau(\mathrm{Y}) \subseteq \mathrm{D}_\tau(\mathrm{X})$*.*

This implies that the only (first order) τ*-types realized in* $EM(X, \mathrm{D}_{\tau'}(\mathrm{Y}))$ *were realized in* M*.*

Let $H(\kappa)$ denote $\beth_{(2^\kappa)^+}$.

The easiest formulation of 'sufficiently long' is:

$|X|$ greater than $H(|\tau|)$.

Tighter estimates are possible, but we don't need them now.

Hanf Numbers

Hanf numbers are functions from cardinals to cardinals.

1. The Hanf number *for omitting* 2^κ *types* is the least cardinal $\Theta(\kappa)$ such that if a first order theory in a vocabulary with cardinal κ has a model of cardinality Θ that omits the family of types then arbitrarily large models of T omit them.

2. The Hanf number *for a logic* $\mathcal{L}$ is the least cardinal $\Theta(\kappa)$ such that if an $\mathcal{L}$-sentence in a vocabulary with cardinal κ has a model of cardinality Θ then it has arbitrarily large models.

3. The Hanf number *for AEC's* is the least cardinal $\Theta(\kappa)$ such that if an AEC in a vocabulary with cardinal κ has a model of cardinality Θ then it has arbitrarily large models.

   In each case, $\Theta(\kappa) \leq H(\kappa) = \beth_{(2^\kappa)^+}$.

We write $H(\tau)$ for $H(|\tau|)$.

EM models for AEC

Theorem 8 *If* $\mathbf{K}$ *is an abstract elementary class in the vocabulary* τ*, which is represented as a* $PC\Gamma$ *class witnessed by* τ', T', Γ *that has arbitrarily large models,* $\mathbf{K}$ *has EM over ordered sets of indiscernibles.*

Thus the Hanf number for having models for the class of AEC s with $|\tau| = \kappa$ is also $H(\kappa)$.

EM models for AEC–Long Form

If $\mathbf{K}$ is an abstract elementary class in the vocabulary τ, which is represented as a $PC\Gamma$ class witnessed by τ', T', Γ that has arbitrarily large models, $\mathbf{K}$ has EM over ordered sets of indiscernibles. there is a τ'-diagram Φ such that for every linear order $(I, <)$ there is a τ'-structure $M = EM(I, \Phi)$ such that:

1. $M \models T'$.
2. The τ'-structure $M = EM(I, \Phi)$ is the Skolem hull of I.
3. I is a set of τ'-indiscernibles in M.
4. $M \restriction \tau$ is in $\mathbf{K}$.
5. If $I' \subset I$ then $EM_\tau(I', \Phi) \preceq_{\mathbf{K}} EM_\tau(I, \Phi)$.

Thus the Hanf number for having models for the class of AEC s with $|\tau| = \kappa$ is also $H(\kappa)$.

Model Homogeneity

Definition 9 *M is μ-model homogenous if for every $N \preceq_{\mathbf{K}} M$ and every $N' \in \mathbf{K}$ with $|N'| < \mu$ and $N \preceq_{\mathbf{K}} N'$ there is a $\mathbf{K}$-embedding of N' into M over N.*

Lemma 10 *(jep) If M_1 and M_2 are μ-model homogenous of cardinality $\mu > \mathrm{LS}(\mathbf{K})$ then $M_1 \approx M_2$.*

Theorem 11 *If $\mathbf{K}$ has the amalgamation property and $\mu*^{<\mu*} = \mu*$ and $\mu* \geq 2^{\mathrm{LS}(\mathbf{K})}$ then there is a model $\mathcal{M}$ of cardinality $\mu*$ which is $\mu*$-model homogeneous.*

A monster model $\mathcal{M}$ exists

GALOIS TYPES

Definition 12 *Let* $M \in \mathbf{K}$*,* $M \preceq_{\mathbf{K}} \mathcal{M}$ *and* $a \in \mathcal{M}$*. The Galois type of* a *over* M *is the orbit of* a *under the automorphisms of* $\mathcal{M}$ *which fix* M*.*

Definition 13 *The set of Galois types over* M *is denoted* $\mathrm{ga-S(M)}$*.*

We say a Galois type p over M is realized in N with $M \preceq_{\mathbf{K}} N \preceq_{\mathbf{K}} \mathcal{M}$ if $p \cap N \neq \emptyset$.

GALOIS SATURATION

Definition 14 *The model M is μ-Galois saturated if for every $N \preceq_{\mathbf{K}} M$ with $|N| < \mu$ and every Galois type p over N, p is realized in M.*

Theorem 15 *For $\lambda > \mathrm{LS}(\mathbf{K})$, the model M is λ-Galois saturated if and only if it is λ-model homogeneous.*

GALOIS STABILITY

Definition 16 *1. Let $N \subset \mathcal{M}$. N is λ-Galois-stable if for every $M \subset N$ with cardinality λ, only λ Galois types over M are realized in N.*

2. $\mathbf{K}$ is λ-Galois-stable if $\mathcal{M}$ is. That is $\mathrm{aut}_M(\mathcal{M})$ has only λ orbits for every $M \subset \mathcal{M}$ with cardinality λ.

CATEGORICITY IMPLIES STABILITY

Theorem 17 *If* $\mathbf{K}$ *is categorical in* λ*, then* $\mathbf{K}$ *is* σ*-Galois-stable for every* $\sigma < \lambda$*.*

This argument has the same form as the first order proof. But one has to choose different linear orders $\lambda^{<\omega}$ for the skeletons.

Corollary 18 *Suppose* $\mathbf{K}$ *is categorical in* λ *and* λ *is regular. The model of power* λ *is saturated and so model homogeneous.*

Proof. Choose in $M_i \preceq_{\mathbf{K}} \mathcal{M}$ using $< \lambda$-stability and Löwenheim-Skolem, for $i < \lambda$ so that each M_i has cardinality $< \lambda$ and M_{i+1} realizes all types over M_i. By regularity, it is easy to check that M_λ is saturated. $\square_{18}$

MORLEY'S METHOD FOR GALOIS TYPES

Lemma 19 *[II.1.5 of Sh394] If $M_0 \leq M$ and*

$$|M| \geq H(|M_0|),$$

we can find an EM-set Φ such that the following hold.

1. *For every I, $M_0 \leq EM(I, \Phi)$.*
2. *$EM_\tau(I, \Phi)$ omits every Galois type over N which is omitted in M.*

Thus the Hanf number for omitting Galois types in an AEC with $|\tau| = \kappa$ is also $H(\kappa)$.

MORLEY'S METHOD FOR GALOIS TYPES–Long form

[II.1.5 of Sh394]
If $M_0 \leq M$ and

$$|M| \geq H(|M_0|),$$

we can find an EM-set Φ such that the following hold.

1. The τ-reduct of the Skolem closure of the empty set is M_0.
2. For every I, $M_0 \leq EM(I, \Phi)$.
3. If I is finite, $EM_\tau(I, \Phi)$ can be embedded in M.
4. $EM_\tau(I, \Phi)$ omits every galois type over N which is omitted in M.

Thus the Hanf number for omitting Galois types in an AEC with $|\tau| = \kappa$ is also $H(\kappa)$.

TOWARDS DOWNWARD CATEGORICITY

Theorem 20 *Suppose $M \in \mathbf{K}$ omits a Galois type p over a submodel M_0 with $|M| \geq H(|M_0|)$. Then there is no regular cardinal $\lambda \geq |M|$ in which $\mathbf{K}$ is categorical.*

Proof. By Lemma 19, there is a model $N \in \mathbf{K}$ with cardinality λ which omits p. But by Lemma 18, the unique model of power λ is saturated. $\square_{20}$

That is categoricity in λ implies all models N are 'log' $|N|$-saturated.

This is analogous to the step in Morley's proof. Every model of λ-categorical theory is $\aleph_1$-saturated.

We need some significant new tools.

TAMENESS

Definition 21 *We say* $\mathbf{K}$ *is* (χ, μ)-weakly tame *if for any saturated* $N \in \mathbf{K}$ *with* $|N| = \mu < \lambda$ *if* $p, q, \in \mathrm{ga-S(N)}$ *and for every* $N_0 \leq N$ *with* $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ *then* $q = p$.

(χ, μ)-tame means we must have the condition for all models not just saturated ones.

We can formulate tameness in terms of groups actions.

Let M be the model-homogenous structure of cardinality μ.

Let $N \preceq_{\mathbf{K}} M$.

If for every $N_0 \preceq_{\mathbf{K}} N$, with $|N_0| \leq \chi$
there is an automorphism of M fixing N_0 and mapping b to b',

then there is an automorphism of M fixing N and mapping b to b',

CATEGORICITY IMPLIES TAMENESS

The following is either a theorem or a conjecture.

Theorem 22 (9.4) *Suppose* $\mathbf{K}$ *is* λ*-categorical for the regular* $\lambda \geq H(\tau)$*. If* $H(\tau) < \chi < \lambda$

Then $\mathbf{K}$ *is* $(\chi, H(\tau))$*-weakly tame.*

$\chi < \lambda$ rather than $\chi \leq \lambda$ is a crucial issue.
See F659 as well as 394.

CRUCIAL CONCEPTS

Vaughtian pairs

Definition 23 *1. A (p, λ) Vaughtian pair is a pair of models $M \preceq_{\mathbf{K}} N$ with p over a submodel of M and p nonalgebraic such that $p(M) = p(N)$ and $|M| = |N| = \lambda$.*

2. A true (p, λ) Vaughtian pair is one where both M and N are saturated.

3. A (p, κ, λ) model is a model N of power κ with $p \in \mathrm{ga-S(M)}$ for some $M \prec N$ and $p(N) = \lambda$; $|M| \leq \lambda$.

Minimal Types

Definition 24 *The Galois type $p \in \mathrm{ga-S(M)}$ is minimal if it nonalgebraic (not realized in M) and for every N with $M \preceq_{\mathbf{K}} N$ and $|N| = |M|$, p has at most one nonalgebraic extension to $\mathrm{ga-S(N)}$.*

DOWNWARD CATEGORICTY

Lemma 25 (9.7*2) *If* $\mathbf{K}$ *is* λ*-categorical for a regular* $\lambda > H(H(\tau))$ *then*

1. *Every model of cardinality* $\geq H(H(\tau))$ *is* $H(H(\tau))$*-saturated.*
2. *Consequently,* $\mathbf{K}$ *is* $H(H(\tau))$*-categorical.*

Lemma 26 (9.7* 6-9) *If* $\mathbf{K}$ *is* λ^+*-categorical for a* $\lambda > H(H(\tau))$ *then there is a model* M^* *with cardinality* $H(\tau)$ *such that there is no* $(p, H(H(\tau)))$ *Vaughtian pair.*

These results use *tameness*, EM-models, Morley's two cardinal theorem and omitting of types.

Now we can work ourselves back up. The limit on going up is the necessary use of tameness.

UP ONE CARDINAL

Theorem 27 *Suppose* $|M| = \mu$, $\mathbf{K}$ *is* μ*-categorical,* $p \in S(M)$ *is minimal, and there is no* (p, μ)*-Vaughtian pair. Then every model of cardinality* μ^+ *is saturated.*

THE INGREDIENTS

Lemma 28 *Let* $p \in \mathrm{ga-S(M)}$, $|M| = \mu$ *and suppose there is no* (p, μ) *Vaughtian pair. Then any* N *with* $|N| = \mu^+$ *and* $M \preceq_{\mathbf{K}} N$ *has* μ^+ *realizations of* p*.*

Definition 29 N *admits a* (p, λ, α)*-resolution over* M *if* $|N| = |M| = \lambda$ *and there is a continuous increasing sequence of models* M_i *with* $M_0 = M$, $M_\alpha = N$ *and a realization of* p *in* $M_{i+1} - M_i$ *for every* i*.*

Theorem 30 *[Grossberg VanDieren]*
Assume $p \in \mathrm{ga-S(M)}$ *is minimal and* $\mathbf{K}$ *does not admit a* (p, λ)*-Vaughtian pair. If* N *admits a* (p, λ, α)*-resolution over* M*, with* $\alpha = \lambda \cdot \alpha$ *then* N *realizes every* $q \in \mathrm{ga-S(M)}$*.*

Proof of UP ONE CARDINAL

The up-one theorem Suppose $|M| = \mu$, $\mathbf{K}$ is μ-categorical, $p \in S(M)$ is minimal, and there is no (p, μ)-Vaughtian pair. Then every model of cardinality μ^+ is saturated.

Proof. Let $N \in \mathbf{K}$ have cardinality μ^+. Choose any $M \preceq_{\mathbf{K}} N$ with cardinality μ. We will show every type over M is realized in N. By μ-categoricity M and p can be taken as in the hypotheses. Fix α with $\alpha \cdot \mu = \mu$. By Lemma 28, p is realized μ^+ times in N; easily, there is N' with $M \preceq_{\mathbf{K}} N' \preceq_{\mathbf{K}} N$ such that N' admits a (p, μ, α)-decomposition. By Theorem 30, N' and *a fortiori* N realize every type over M and we finish. $\square_{27}$

SPLITTING-SHORT FORM

There is a notion of non-splitting, patterned on the first order case, which has appropriate extension properties to yield.

Suppose an AEC with amalgamation and joint embedding is categorical in λ. If $M \preceq_{\mathbf{K}} N$ are saturated models in $\mathbf{K}$ with $|M| < |N| \leq \lambda$ and $p \in \mathrm{ga}-\mathrm{S(M)}$, p has an extension to $\mathrm{ga}-\mathrm{S(N)}$ which does not $|M|$-split over M.

The proof requires a careful use of EM models and properly chosen linear orders.

GOING UP

Suppose $(H(\tau), \infty)$-tame and **K** is λ^+-categorical for some $\lambda > H(H(\tau)) = H_2$. Then **K** is categorical in all cardinals greater than H_2.

1. The hypothesis of 'up-one' are satisfied by the model in H_2. So **K** is both H_2 and H_2^+-categorical.
2. (Grossberg-VanDieren). For any $\mu > H_2$, categoricity in μ and μ^+ implies categoricity in μ^{++}.
3. (Grossberg VanDieren) If κ is singular and there is a unique model in each cardinality less than κ, which is saturated, then every model of power κ is saturated. Hence, **K** is κ-categorical.
4. In fact, this model of power κ also satisfies the hypotheses of Theorem up-one. (Using non-splitting to transfer Vaughtian pairs and minimal types.)

SUMMARY

Working in AEC with amalgamation, we can use four important tools:

1. Galois-types
2. EM models over indiscernibles
3. Vaughtian pairs
4. splitting

If we add the full tameness hypothesis, this resolves the eventual categoricity conjecture.

Problems:

1. Remove tameness hypothesis
2. Remove amalgamation hypothesis
3. Make connections with syntax – e.g. $L_{\omega_1,\omega}(Q)$.

References

[1] J.T. Baldwin. Ehrenfreuht-Mostowski models in abstract elementary classes. to appear, 200?

[2] J.T. Baldwin. Notes on aec. Available at www.math.uic.edu/ jbaldwin, 200?

[3] R. Grossberg and M. Van Dieren. Upward categoricity transfer theorem for abstract elementary classes. preprint.

[4] S. Shelah. Categoricity for abstract classes with amalgamation. *Annals of Pure and Applied Logic*, 98:261–294, 1999. paper 394.