Zilber's notion of logically perfect structure: Universal Covers

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Abstract

We sketch the mathematical back ground and the main ideas in the proofs of categoricity of theories of several examples of universal covers – reducing an analytic to a model theoretic (discrete) description. We hope this discussion will be useful to a wide spectrum of mathematicians ranging from those working in geometry to those working in logic; specifically, model theory.

Contents

1 Introduction

The goal of this paper is to sketch (hopefully for a wide spectrum of mathematicians ranging from those working in geometry to those working in logic; specifically, model theory) some recent interactions between model theory and a roughly 150-year old study of analytic functions involving complex analysis, algebraic topology, and number theory that explore the canonicity of universal covers. Towards this goal we discuss and present several examples indicating the main ideas of the proofs and the necessary changes in method for different situations.

Here is Zilber's description of his own project (from his 2000 Logic Colloquium talk in Paris [Zil05a]):

> *The initial hope of this author in [Zil84] that any uncountably categorical structure comes from a classical context (the trichotomy conjecture),*

was based on the belief that logically perfect structures could not be overlooked in the natural progression of mathematics. Allowing some philosophical license here, this was also a belief in a strong logical predetermination of basic mathematical structures. As a matter of fact, it turned out to be true in many cases. ...Another situation where this principle works is the context of o-minimal structures [PS98].

A rather ambitious project aimed at finding *categorical* axiomatizations (Definition 3.0.1) of various kinds of *universal covers* has been unfolding in the 21st century. The simplest example of such universal covers is given by the short exact sequence:

$$0 \to \ker(\exp) \to (\mathbb{C}, +, 0) \stackrel{\exp}{\to} (\mathbb{C}, +, \cdot, 0, 1) \to 1. \tag{1}$$

Zilber's original project really aimed to understand the sequence

$$0 \to \ker(\exp) \rightarrowtail (\mathbb{C}, +, \cdot, \exp) \stackrel{\exp}{\twoheadrightarrow} (\mathbb{C}, +, \cdot, \exp) \to 1. \tag{2}$$

The first diagram describes a two-sorted *cover* of the multiplicative group by the additive group. The *full* field structure is studied on the range space although the kernel is of the homomorphism from $(\mathbb{C}, +, 0)$ to $(\mathbb{C}, \cdot, 1)$.

The second [Zil04] corresponds to the *theory* of the complex exponential field. The domain and range of the map are the same exponential field but the kernel is again computed with respect to the homomorphism $\exp$ from $(\mathbb{C}, +) \to (\mathbb{C}^*, \times)$.

In both cases, first order axioms are supplemented by an $L_{\omega_1,\omega}$-sentence asserting the kernel is isomorphic to $\mathbb{Z}$, i.e., is standard. Here, we focus on three main *families* of generalizations (described in the chart below) of the first diagram. As this question was extended to more general algebraic contexts, the fundamental cover diagram from equation (1) changed to this more general situation:

$$C \stackrel{p}{\twoheadrightarrow} S(\mathbb{C}). \tag{3}$$

Notice two things:

- The map p remains a projection, but it will significantly change as the family of examples unfolds. Also,
- there is no longer a kernel when $S(\mathbb{C})$ is not a group.

Therefore, in a rather Protean way, the infinitary description that in the particular case described a 'standard kernel' assumes various guises for different examples. Usually, the descriptions are of 'standard fibres' rather than having a 'standard kernel'.

Crucially, in all cases except part of § 5 the target will be some kind of definable set in an algebraically closed field. The necessary vocabulary for the domain will vary among the situations considered. Shimura varieties require a more general domain:

Notation 1.0.1. *(The general situation)*

$$X^+ \xrightarrow{p} S(\mathbb{C}) \to 1. \tag{4}$$

Here, $S(\mathbb{C})$ is a variety arising as the quotient of the action of a discrete group on $\mathbb{H}$ (hyperbolic space) or more generally (Shimura varieties) on a hermitian symmetric domain X^+. The target is described by a first order theory $T := \mathrm{Th}(S(\mathbb{C}))$ in a large enough (field) countable vocabulary with quantifier elimination (possible, as S is definable in $(\mathbb{C}, +, \times)$). Notation 1.0.1 thus instantiates the general schema, with appropriate notations for specific cases to be given as we discuss them. Zilber describes the value of his project in terms of 'a complete formal invariant' (Remark 5.3.2).

> *The geometric value of the project is perhaps in the fact that the formulation of the categorical theory of the universal cover of a variety X ... is essentially a formulation of a complete formal invariant of X.*
>
> [DZ22b, 1]

The following chart organizes the papers which are the major source for this study. It also provides a keyword describing the main method or context used, and the section of this paper where issues around the specific variant are explained.

	topic	paper	method/context	section
1	Complex exponentiation	[Zil05b]	quasiminimality	§1
2	cov mult group	[Zil06]	quasiminimality	§1
3		[BZ11]	quasiminimality	
4	j-function	[Har14]	background	§4.1
5	Modular/Shimura Curves	[DH17]	quasiminimality	§4
6	Modular/Shimura Curves	[DZ22b]	quasiminimality	
7	finite Morley rank groups	[BGH14]	fmr & notop	§5.1
8	Abelian Varieties	[BHP20]	fmr & notop /quasiminimality	§5.3
9	Shimura *varieties*	[Ete22]	notop	§6
10	Smooth varieties	[Zil22]	o-quasiminimality	§8

In this chart, the first line [Zil04] (an axiomatization of the exponential map from the complex field to itself) differs from the others in the role of the quantifier 'there exists uncountably many'. In that case it is essential to directly control the *cardinality of the algebraic closure of a countable set*. Moreover, in line 1 the domain has a field structure that disappears in the two-sorted approach of the rest. In the other lines of the chart, the infinitary logic $L_{\omega_1,\omega}$ is used to control the size of fibers of the cover or when the structure is a group the size of the kernel. This requirement suffices to also control the cardinality of the algebraic closure. Lines 2-6 deal with *curves* (1-dimensional objects) where categoricity is obtained by quasiminimality. The third big horizontal block deals with higher dimensional varieties. Lines 7 and 9 stray from formal categoricity towards more traditional descriptions of models; quasiminimality is replaced by a different version of excellence arising in Shelah's study of notop theories (an important notion in Classification Theory). Both quasiminimality and 'notop' apply to line 8. The last line considers families of covers of arbitrary smooth algebraic varieties with an infinitary logic construction defined over o-minimal expansions of the reals. There, the focus is on categoricity in $\aleph_1$.

It is worth noting that we could have organized our chart under a totally different scheme. The Abelian varieties and $(\mathbb{C}, +)$ are specific varieties. The j-function and the Shimura varieties may be regarded as moduli spaces for (generalized) families of varieties[1]. After preliminary discussions on the model theoretic framework, in Section 4 we sketch in some detail categoricity of universal covers of modular curves. In the later sections we describe the modifications to this program necessary for higher dimensions.

1.1 Mathematical Encounters

1.1.1 Some ancient history: In and out of the Zilber world

The first author turns to the first person singular for some memories:

Zilber and I both received our Ph.D.'s in the early 1970's. An important result appeared in both theses: the solution to Morley's conjecture that an $\aleph_1$-categorical theory has finite Morley rank. Such an overlap was not an issue during the Cold War. (On the other hand, Baldwin's advisor, Lachlan, had to write an entirely new thesis when the result of the proposed one appeared in the west as he was about to submit.)

I first (given my zero knowledge of Russian) learned in any detail of Zilber's work during the 1980-81 model theory year in Jerusalem. Greg Cherlin had no such deficiency and gave with Harrington and Lachlan an alternate proof of Zilber's theorem that there were no finitely axiomatizable totally categorical theories. They relied on the classification of finite simple groups. A few years later Boris completed his model theoretic proof of the key combinatorial lemma avoiding that reliance.

I first knew Boris in any depth during the model theory semester in Chicago 91-92. Unfortunately, I had partially financed a semester by agreeing to be acting head the Fall semester, thereby restricting my mathematical activity. In that busy fall, Boris and Angus Macintyre lectured on Tuesday's on Zariski geometries and o-minimality,

[1] Various types of Shimura varieties include Siegel, PEL-type, and Hodge-type; only some parameterize algebraic avarieties.

respectively. The lively group include Macintyre, Zilber, Laskowski, Marker, Otero, D'Aquino and myself, with Pillay driving in weekly from Notre Dame. Lunch was at a deli that Boris insisted on because of the soup followed by coffee at Jamoch's, the first modern coffee house in the UIC area.

About that time, I began work on the Hrushovski construction, but in a quite different direction from Boris: predimension with irrational α. This led to my work with Shelah giving the first full proof of the $0-1$ law with edge probability $n^{-\alpha}$ and that the theory of the Shelah-Spencer graph was stable, building on the 1992 Ph.D. thesis of my student Shi. And this led to work with Kitty Holland on fusions, giving the first construction of a rank 2 field with a definable infinite predicate. And then back to Boris and his work on complex exponentiation. Understanding his notion of quasiminimal excellence inspired the desire to understand Shelah's more general notion of excellence. Thence came my monograph on abstract elementary classes and subsequent work on infinitary logic. In any case, visits several times a decade to Oxford always were exciting sources of ideas and pleasant times.

1.1.2 An unlikely encounter of two areas: MAMLS at Rutgers, 2001

The second author of this paper witnessed and participated in one of those momentous encounters of two areas that only seldom happen: during the MAMLS Meeting at Rutgers in February 2001, a group of people working in Abstract Elementary Classes (including Rami Grossberg, Monica VanDieren, Olivier Lessmann and the second author of this paper) was very busy discussing Shelah's notion of excellence, originally linked to his work in the model theory of $L_{\omega_1,\omega}$. The *n-amalgamation diagram* was very much part of that discussion. There was a lecture by Boris Zilber at the end of the day, and we all attended, not expecting to understand much, but eager to see him speak. To our great surprise, at the end of Zilber's lecture (dealing with exponential covers, mentioning many analytic number theoretic methods that were arcane to us, and mixing in areas such as "Nevanlinna Theory", etc.), he asked a final question and drew a picture underscoring his question. Boris's picture was *exactly* the n-amalgamation diagram we had been discussing thoroughly with the AEC people those very same days; Boris's question was exactly about the behaviour of types in the amalgam and how it could be controlled by small pieces in the components. We jumped to talk to him at the end of his lecture, with the excitement of seeing a potential connection. Boris said he didn't know the model theory of $L_{\omega_1\omega}$ but he would look into excellence...

The rest is history: after a few weeks, a first draft of a proof of properties of pseudoexponentiation drawing on a version of excellence and quasiminimality in $L_{\omega_1\omega}$ was circulated, and Zilber started using many methods from excellent classes and infinitary logic. The richness of this approach has provided many interesting connections; we explore some of them in our paper.

1.2 A word of thanks from the second author

Here, the second author turns to the first person singular, for this excerpt:

> *I would like to thank Boris Zilber, at a very personal level, for a life-changing conversation we had in 2007 in Utrecht, during a meeting organized by Juliette Kennedy, on connections between Mathematics, Philosophy and Art. One evening, after dinner, Boris said "let's go for a walk and speak a bit about mathematics." In the cold night along the canals, he described, for about an hour, some of what he had been doing—I kept asking and asking questions. At some point, on a bridge, he turned to me and said: "But you, in what have you been working?" I tried to gather my thoughts on the spot while walking, and started describing a project we had back then, with Berenstein and Hyttinen [BHV18], of understanding independence notions in continuous logic, trying to extend the work of Chatzidakis and Hrushovski to the continuous case, and encountering difficulties. Boris asked me to describe briefly continuous model theory and continuous abstract elementary classes. At some point, he said I obviously had tools for dealing with model theoretical approaches to quantum mechanics. I asked how so. He said "look at Gelfand triples, ... ". I returned to Helsinki where I was spending a sabbatical, and Boris's remarks made a deep change in my own approach to model theory, in the possibilities I started slowly unfolding. I am deeply grateful for that momentous conversation, and for all the lines of work that have derived from that evening!*
>
> Andrés Villaveces

The authors want to thank many people who helped this project go through. Among them, hoping not to forget important people, are, most notably Sebastián Eterović, Jim Freitag, Jonathan Kirby, Anatoli Libgober, Ronnie Nagloo, and Boris Zilber. Without their attention to our discussions, online, at conferences, and on campus, this project would have been much harder to complete. The first author especially wants to thank Ronnie and Sebastián for hours of conversation. The second author especially thanks Alex Cruz and Leonardo Cano for many helpful discussions related to these subjects in the Bogotá seminar before this project started. Finally, discussions with Thomas Kucera and Martin Bays were very important at earlier stages of the construction of this paper. Finally, the referee reports were invaluable.

2 Model theory in Mathematics

We first deal with some variations in model theoretic and geometric terminology.

2.1 Model theoretic background

Mathematical logic makes a central distinction between a vocabulary and a collection of sentences in a logic. For this reason, we use 'language' only for the second and reserve 'vocabulary' for what is sometimes called similarity type.

Definition 2.1.1 (Vocabulary and Structure)**.** *1. A vocabulary τ is a collection of constant, relation, and function symbols (with finitely many arguments).*

2. A τ -structure is a set in which each τ-symbol is interpreted, e.g., an n-ary relation symbol as an n-ary relation.

Definition 2.1.2. Full formalization *involves the following components.*

1. *A **vocabulary** with associated notion of structure as in Definition 2.1.1.*
2. *A **logic** $\mathcal{L}$ has:*
   - **a** *A class $\mathcal{L}(\tau)$ of 'well formed' **formulas**.*
   - **b** *A notion of '**truth** of a formula' from the class $\mathcal{L}(\tau)$ in a τ-structure, usually denoted $\mathfrak{A} \models \varphi$.*
   - **c** *A notion of a "formal **deduction**" for this logic.*
3. ***Axioms**: Specific sentences of the logic that specify the basic properties of the situation in question.*

Example 2.1.3. *(Three important logics.)*

1. *The **first order language** $\mathcal{L}_{\omega,\omega}(\tau)$ associated with τ is the least set of formulas containing the atomic τ -formulas and closed under* finite *Boolean operations and quantification over finitely many individuals.*
2. *The $\mathcal{L}_{\omega_1,\omega}(\tau)$ **language** associated with τ is the least set of formulas containing the atomic τ -formulas and closed under* countable *Boolean operations and quantification over finitely many individuals.*
3. *The **second order language** associated with τ, denoted $\mathcal{L}^2(\tau)$, is the least set of formulas extending $\mathcal{L}_{\omega,\omega}(\tau)$ by allowing quantification over sets and relations. $\mathcal{L}^2(\{=\})$ is symbiotic ('morally equivalent', roughly speaking) with set theory.*

Morley rank (corresponding to the Krull/Weil dimension in the particular case of fields) was introduced in [Mor65] to study theories categorical in uncountable power. Section 5 explores the role of finite Morley rank groups in studying covers. Three good sources for the more advanced model theory used here are [Mar02, TZ12, Poi85].

2.2 Various Viewpoints

We now discuss two quite different uses of the three words *automorphism*, *model* and *definable*, coming from areas of mathematics relevant to this paper. (The difference in use depending on the area of mathematics has been at times a source of confusion.)

Remark 2.2.1. (Automorphism: two notions)

In Model Theory: *An* ***automorphism*** *of a τ-structure $\mathcal{A}$ is a permutation of its universe A that preserves (in both directions) each relation or function symbol for τ. For instance, the automorphisms of a geometry (when given in terms of lines and points together with an incidence relation) are the* collineations.

In Algebraic Geometry: *An* ***automorphism*** *of a variety is an invertible morphism*[2].

Remark 2.2.2. (Model: two notions)

In Model Theory: The word *model* also sees different uses depending on the area. In logic, a model is sometimes just a τ-structure but often signifies that the structure satisfies a theory (as in '$(\mathbb{C}, +, \cdot, 0, 1)$ is a model of the theory ACF_0'). *Minimal model* might mean 'no proper elementary submodel' or, very differently, 'every definable subset is finite or cofinite'.

In Algebraic Geometry: A **model** is a specific biregularity class within a birational equivalence class. In Weil/Zariski style, a variety is determined by a coordinate ring, but only up to isomorphism of this coordinate ring. A 'model' of the variety might be a specific affine variety with that coordinate ring, but any biregularly isomorphic variety would also be a model.

Thus, unlike model theory, algebraic geometry does not identify 'models' up to isomorphism. Rather, it looks for a specific 'canonical representation' among 'isomorphic solution sets'. A *minimal model* is a smooth variety X with function field K such that if Y is another smooth variety with function field K and $f\colon X \mapsto Y$ is birational, then f is an isomorphism.

Remark 2.2.3. (Definable/defined: two notions)

In Model Theory: A subset X of a model M^n is *defined* over a set A if there is a formula $\phi(\mathbf{x}, \mathbf{a})$ with solution set X.

In usual mathematics the word 'defined' is often short for 'well-defined' saying that the value of a function defined on a quotient space does not depend on the choice of a representative.

[2] This begs the question of defining morphism. A good approximation is 'definable map'. In algebraic geometry a morphism is (cf [Poi87, p 79: section 4.4]) a constructible (generically quasi-rational) bijection. Biregular and birational are more specific syntactic restrictions on an isomorphism.

In model theory, we add the adjective ‘definable’ when there is a formula of the language that captures the notion. Thus, the algebraic geometric ‘automorphism’ becomes ‘definable bijection’. It is worth noting that many important automorphisms in algebraic geometry do not necessarily preserve structure.

Remark 2.2.4 (Why infinitary logic?)**.** A natural question at this point is: Why is axiomatizability in $L_{\omega_1,\omega}$ relevant to geometric questions? The answer to this question is not univocal, and strongly reflects different historical issues arising in different areas of mathematics. We discuss four responses, two from ordinary mathematics, two from logic.

1. In ordinary mathematics:
   (a) The constraints of expressibility offered by a particular logic force a detailed analysis of the hypotheses of a result. This analysis in similar earlier cases has led to, for example, the Zilber-Pink conjecture and the Conjecture on the Intersection of Tori (see e.g. [BMPTW20]).
   (b) Of course, each of the ‘canonical structures’ is explicitly definable in set theory. But this definition in most cases is useless for studying the object. Useful succinct second order axioms are available for the real and complex numbers but are only partially known for universal covers. First order logic is stymied *a priori* by the intractability of arithmetic. Thus, categoricity in infinitary logic is essential for *giving an ‘algebraic’ account of an ‘analytic object’*. This use of model theory can be seen as part of the larger scale *GAGA* mathematical program of bridging analytical concepts and algebraic ones.
2. In logic (in particular, in model theory):
   (a) A natural question is: are there important mathematical notions expressible in infinitary logic which are not expressible in first order? The study of complex exponentiation yielded a superb initial example: the categoricity of the covering map of $\mathbb{C}^*$ in [BaysZil].
   (b) This raises the question of what are the *new* axioms in this paper that require an infinitary description. The infinite dimension axioms are well known and the switch from ‘standard kernel’ to ‘standard fiber over z’ (i.e. $q^{-1}(z)$) is unremarkable. It seems the finite index conditions (Section 4.4) are not first order expressible.

3 Categoricity, quasiminimality and excellence

We give a quick sketch of notions around categoricity[3] and the history of their logical development.

Definition 3.0.1 (Categoricity). *1 A* theory T *in a logic* $\mathcal{L}$ *is a collection of* $\mathcal{L}$*-sentences in a vocabulary* τ.

2 T *is* categorical in cardinality κ *(*κ*-categorical) if all models* M *of* T *with* $|M| = \kappa$ *are isomorphic.*

Although certain canonical mathematical structures are fruitfully axiomatized in second order logic, rather than second order categoricity, we usually consider these characterizations as defining these structures *in set theory.* Such definitions are exactly what it means to be a structure. Second order categoricity *per se* gives no useful mathematical information. In contrast, κ-categoricity in first order logic or in $L_{\omega_1,\omega}$ provides very significant (combinatorial geometric) information; it assigns a dimension to each model.

3.1 The Classical Categoricity Theorems

The following results survey the spectrum of cardinals in which certain types of theory can be categorical. These theorems are of the form *if a theory (or a sentence) is categorical in some high enough cardinal(s), then it must be categorical on a tail of cardinals.*

Theorem 3.1.1 (Morley's Categoricity Theorem). *A countable first order theory is categorical in one uncountable cardinal if and only if it is categorical in all uncountable cardinals. [Mor65].*

Theorem 3.1.2 (Shelah's Categoricity under the weak continuum hypothesis below $\aleph_\omega$). *Assuming* $2^{\aleph_n} < 2^{\aleph_{n+1}}$ *a* sentence *in* $L_{\omega_1,\omega}$ *that is categorical in* $\aleph_n$ *(for every* $n < \omega$*) is categorical in all uncountable cardinals [She83a], [She83b].*

Theorem 3.1.3 (Shelah's Categoricity theorem for excellent sentences). *An* excellent sentence *in* $L_{\omega_1,\omega}$ *is categorical in one uncountable cardinal if and only if it is categorical in all uncountable cardinals [She83a], [She83b].*

Theorem 3.1.4 (Zilber's Categoricity for quasi-minimal excellent classes). *A quasi-minimal excellent class is categorical in all uncountable cardinals [Zil04].*

[3]More specifically, when in model theory we use the word *categoricity,* we mean categoricity in a specific cardinality or 'in power'. See a thorough discussion of categoricity in various logics in [Bal18, §3.1] and an exposition of the philosophical import of the notion in [CMVZ21].

3.2 Pregeometries (matroids) and quasiminimality

The presence of quasiminimal pregeometries provides an extremely fruitful and natural control of models in a class (and of their interactions).

Definition 3.2.1 (Combinatorial Geometry)**.** *A* closure system *is a set G together with a 'closure' relation on subsets of G*

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \ X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) *is a* pregeometry *if, in addition, we have:*
A4. *If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.*

If points are closed ($cl(\{a\}) = \{a\}$, for each a) the structure is called a geometry.

Pregeometries are virtually the same mathematical objects as matroids.

Definition 3.2.2. *1. A subset D of a τ-structure M is* ***first order-definable*** *in M if there is $\mathbf{a} \in M$ and an $\mathcal{L}_{\omega,\omega}(\tau)$-formula $\varphi(x, \mathbf{y})$ such that $D = \{m \in M : M \models \varphi(m, \mathbf{a})\}$. If $\mathbf{a} \in A \subseteq M$, D is definable with parameters from A.*

2. $\mathrm{acl}_M(A)$ *(the* algebraic closure *of A in M) is $\{m \in M : \phi(m, \overline{a}), \overline{a} \in A\}$, where $\phi(x, \overline{a})$ has only finitely many solutions in M.*

3. $\mathrm{dcl}_M(A)$ *(the* definable closure *of A in M) is defined as was the algebraic closure, but replacing 'finitely many' by 'one'.*

4. *An infinite definable subset D (or its defining formula $\varphi(x)$) is* strongly minimal *if every definable subset of D in every elementary extension of M is finite or cofinite.*

5. *A theory is* strongly minimal *if the formula $x = x$ is strong minimal.*

The notion of type is a crucial tool in model theory.

Definition 3.2.3. *1. The* ***first order type*** *of a over B (in M), denoted $\mathrm{tp}_M(a/B)$, is the set of $\mathcal{L}_{\omega,\omega}$-formulas with parameters from B that are satisfied in M (for $a, B \subseteq M$).*

2. *The* ***quantifier-free type*** *of a over B (in M), denoted $\mathrm{tp}_{\mathrm{qf}}(a/B \colon M)$, is the set of* quantifier-free *first order formulas $\varphi(x, \mathbf{b})$ such that $M \models \varphi(a, \mathbf{b})$ (as before, $\mathbf{b}$ ranges over tuples of B).*

In most contexts, when we just say 'the type of a over B,' we mean the first order type. Note also that if a property is defined without parameters in M, then it is uniformly defined in all models of $\mathrm{Th}(M)$ (the *theory* of M, i.e., the set of all τ sentences that are true in M).

Here are three fundamental observations on strongly minimal sets.

- A strongly minimal set admits a combinatorial geometry when the closure is taken as acl (Definition 3.2.2).
- There is a unique type of elements in a strongly minimal set that are not algebraic. This is called the *generic type* for D.
- In many important examples (e.g. DCF_0), the structure of the model is controlled by its strongly minimal sets.

Shelah's abstract notion of independence (for some first order theories, crystallized as *non-forking*) weakens the notion of combinatorial geometry by dropping **A3**; in some desirable cases this property is recovered on the points realizing a *regular type* and in even better cases the dimensions of the regular types determine the isomorphism type of the model. However, *a priori,* the existence of a global dimension is unusual.

We now look at the generalization of strong minimality, introduced by Zilber, that is central in the connections between model theory and algebraic geometry described in this paper.

Definition 3.2.4 (Quasiminimal structure)**.** *A structure M is* quasiminimal *if every first order ($L_{\omega_1,\omega}$) definable subset of M is countable or cocountable. Algebraic closure is generalized by saying $b \in \mathrm{acl}'(X)$ if there is a first order formula with* **countably many** *solutions over X which is satisfied by b.*

Definition 3.2.5 (Quasiminimal excellent geometry)**.** *Let K be a class of L-structures such that $M \in K$ admits a closure relation cl_M mapping $X \subseteq M$ to $\mathrm{cl}_M(X) \subseteq M$ that satisfies the following properties.*

1. ***Basic Conditions***
   (a) *Each cl_M defines a pregeometry on M.*
   (b) *For each $X \subseteq M$, $\mathrm{cl}_M(X) \in K$.*
   (c) *countable closure property (ccp): If $|X| \leq \aleph_0$ then $|\mathrm{cl}(X)| \leq \aleph_0$.*
2. ***Homogeneity***
   (a) *A class K of models has $\aleph_0$-***homogeneity over** $\emptyset$ *(Definition 3.2.5) if the models of K are pairwise qf-back and forth equivalent (Definition 4.3.7)*

(b) *A class K of models has $\aleph_0$-**homogeneity over models** if for any $G \in K$ with G empty or a countable member of K, any H, H' with $G \leq H, G \leq H'$, H is qf-back and forth equivalent with H' over G.*

3. *K is an* almost quasiminimal excellent geometry *if the universe of any model $H \in K$ is in* $\mathrm{cl}(X)$ *for any maximal* cl-*independent set $X \subseteq H$.*

4. *We call a class which satisfies these conditions an* almost quasiminimal excellent geometry *[BHH+14].*

An almost quasiminimal excellent geometry with strong submodel taken as $A \leq M$, if $\mathrm{acl}_M(A) = A$, gives an *abstract elementary class* (AEC)[4]. But the distinct notion of a quasiminimal AEC (defined in terms of $\leq$ rather than any axioms) is due to [Vas18].

To obtain that the class is complete for $L_{\omega_1,\omega}$, [Kir10, BHH+14] add the requirement of $\aleph_0$-categoricity.

Remark 3.2.6. This definition differs only superficially from those in e.g. [Kir10], where the connections with the combinatorial geometry was emphasized by distinguishing the treatment of elements depending on whether they were in $\mathrm{cl}(H)$. **However,** [BHH+14] required a quasiminimal structure to have a unique generic type. This requirement fails in the two-sorted treatment we deal with here; there may be acl-bases in each sort. So we replace quasiminimality with *almost quasiminimality* (less explicit in [BHP20]) and we thus restore Zilber's first intuition (Definition 3.2.4) that quasiminimality means that all definable sets are countable or co-countable.

Remark 3.2.7 (Excellence)**.** From Zilber's introduction of the notion in [Zil04], it has been known that the axioms 3.2.5 imply $\aleph_1$-categoricity. See the exposition in [Bal09]. But, without further 'excellence' hypotheses, it was unknown whether the class had larger models. Two formulations of excellence are 1) [She83a, She83b]: n-amalgamation of independent systems of models, for all $n < \omega$, and 2) A local condition on the properties of a 'crown' [Kir10]. Either of these implies the existence of arbitrarily large models for theories in $L_{\omega_1,\omega}$. As we discuss in Section 5.2, influenced by work Hart and Shelah on first order classification theory, the next result (here modified by 'almost') clarified the relationship.

Remark 3.2.8. *Crucial Fact [Theorem: Bays, Hart, Hyttinen, Kesälä, Kirby]. Every almost-quasiminimal class (Definition 3.2.5) is excellent as described in Remark 3.2.7. Thus, it is categorical in all uncountable cardinalities.*

[4]See [GL02] for the early history of the model theory of AECs.

4 Modular and Shimura Curves

We begin with an astronaut's view of the j-function and then turn to the model theoretic treatment of some generalizations.

4.1 The great confluence

The general form (over a field of characteristic 0) of an elliptic curve is

$$y^2 = x^3 + ax + b.$$

At least since Diophantus (3rd century AD), the search for integer solutions for such equations has been a central question. The cataloguing of such equations was a major achievement of the 19th century. One key step toward this classification is to generalize the original problem and look first for complex solutions. The solution set of an elliptic curve is then a smooth, projective, algebraic curve of genus one. It can be thought of as a 'classical torus' $\mathbb{T}_\tau := \mathbb{C}/\Lambda_\tau$, where $\tau \in \mathbb{C}$ and Λ_τ is the lattice in $\mathbb{C}$ (the subgroup of $(\mathbb{C}, +)$ generated by $\langle 1, \tau \rangle$.

Klein studied modular and automorphic functions, which provide surprising and deep links between geometry, complex analysis and number theory. The most famous example is the j-function, analytic on $\mathbb{H} = \{z : \mathrm{im}(z) > 0\}$, the upper half plane, and maps onto $\mathbb{C}$ and meromorphic with some poles on the real axis and the following remarkable properties.

Theorem 4.1.1 (Classification of tori by the j-function)**.** *The following are equivalent:*

1. *There exists* $s = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ *such that* $s(\tau) = \frac{a\tau+b}{c\tau+d} = \tau'$,
2. $\mathbb{T}_\tau \approx \mathbb{T}_{\tau'}$ *in the algebraic geometry sense of Definition 2.2.1.*
3. $j(\tau) = j(\tau')$

This rather astonishing classical fact paves the way toward modern day classifications. It provides equivalences between analytic and number-theoretic notions. Strikingly, j is defined as a rational function of two analytic functions g_2 and g_3 (each of them coding so-called 'modularity' properties):

$$j(\tau) = 12^3 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^3}.$$

But where does the word 'elliptic' come from? A meromorphic function is called an *elliptic function*, if it is doubly periodic: there are two $\mathbb{R}$ -linear independent complex numbers ω_1 and ω_2 such that $\forall z \in \mathbb{C}$, $f(z + \omega_1) = f(z)$ and $f(z + \omega_2) = f(z)$.

Abel discovered such doubly periodic functions arose from the solutions of elliptic integrals – originally defined to find the arc length of an ellipse. Weierstraß used the symbol $\wp$ to denote a family of functions $\wp(z, \Lambda_\tau)$ where the defining double sum runs over the elements of the lattice Λ_τ, generated by 1 and τ. The crucial property of the function is that every meromorphic function that is periodic on Λ_τ is a rational combination of $\wp(z, \Lambda_\tau)$ and $\wp'(z, \Lambda_\tau)$. This field of functions is precisely Abel's field of elliptic functions.

Klein's discovery of the j function unified the results of Weierstraß. In his famous investigation of the psychology of mathematical investigation, Hadamard devotes several pages to Poincaré's generalization of the j-function to the family of functions derived from Fuchsian group actions. The crucial phrase for us is *'the transformations I had used to define the Fuchsian functions were identical with those of non-Euclidean geometry'* [Had54, p 33].

This completes a very quick summary of the 19th century predecessors of the theory of moduli spaces, developed in the next section. This study involves complex analysis, actions by a discrete group, number theory, and non-Euclidean geometry. The crucial model theoretic step is to formalize in a vocabulary for two-sorted structures of the form

$$\mathfrak{A} = \langle \langle H; \{g_i\}_{i\in\mathbb{N}} \rangle, \langle F, +, \cdot, 0, 1\rangle, j : H \to F \rangle$$

where $\langle F, +, \cdot, 0, 1\rangle$ is an algebraically closed field of characteristic 0, $\langle H; \{g_i\}_{i<\omega}\rangle$ is a set together with countably many unary function symbols, and $j\colon H \to F$.

In the next section we provide some of the mathematical background for a formal analysis of these two-sorted structures.

4.2 Moduli Spaces

Moduli spaces in geometry are parametrized collections of objects, together with equivalences that allow us to see when two objects are in some sense 'the same', and with families that articulate the variation between the objects in the collection. Paraphrasing the important survey [BZ08], 'moduli spaces are a *geometric* solution to a geometric classification problem.' They parametrize collections of geometric *objects*, they define *equivalences* to say when two objects are the 'same', and establish *families* that determine how we allow our objects to vary or modulate.

In model theory, the notion of a uniform family of definable sets has been thoroughly studied. Such a family is given by a formula of the form $\phi(\mathbf{x}, \mathbf{y})$. Each set in the family is the solution set of $\phi(\mathbf{a}, \mathbf{y})$ (for some $\mathbf{a}$), and the set $\{\mathbf{a} : (\exists \mathbf{y})\phi(\mathbf{a}, \mathbf{y})\}$ is an indexing set of the family. In the algebraic geometry setting, one can require that the $\mathbf{x}$ fall into a variety V and the $\mathbf{y}$ into a variety $W_{\mathbf{a}}$. V is a step toward the notion of a *moduli space*.

Except in § 5, we consider moduli spaces arising from a pair (G, X) consisting of a group G acting on a space X. The algebraic varieties we study arise as quotients $\Gamma \setminus X$ (for Γ a subgroup of G, see Definition 4.2.2). A modular curve arises as a connected component of quotient of $\mathbb{H}$ by congruence subgroups (Definition 4.2.9) of $\mathrm{GL}_2(\mathbb{R})$. Shimura generalized the topic to groups acting on wider classes of domains. Shimura curves are rather more complicated yet generally share similar categoricity properties. Shimura varieties of higher dimension raise many new issues that we sketch in Section 6. In this section, we consider only covers of modular curves by $\mathbb{H}$.

Here, $\mathbb{H}$ refers, as in the rest of this paper, to the upper half complex plane (as the set of points: $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$). $\mathbb{H}$ is also called the hyperbolic plane (when endowed with a metric and topology that make it hyperbolic rather than Euclidean). See [Miy89] for a detailed description. In all our examples, the function p maps the hyperbolic plane into a complex variety. We consider the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ as fractional linear transformations: for
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$, $A(\tau) = \left(\frac{a\tau+b}{c\tau+d}\right)$.

The group of bijections (isometries, $\mathrm{isom}(\mathbb{H})$) that preserve the hyperbolic metric of $\mathbb{H}$ is generated by $\mathrm{PSL}_2(\mathbb{R})$ and the map $z \mapsto -\overline{z}$; $\mathrm{PSL}_2(\mathbb{R})$ consists precisely of all those isometries that preserve orientation (e.g. [Kat92]). After outlining here the classical theory of such actions and moduli spaces, in section 4.3 we describe a model theoretic approach.

Definition 4.2.1 (Fuchsian group)**.**

1. *A subgroup $G \leq \mathrm{isom}(\mathbb{H}) \approx \mathrm{PSL}_2(\mathbb{R})$ is* discrete *if it is discrete in the induced topology.*
2. *A* Fuchsian group *is a discrete subgroup of* $\mathrm{PSL}_2(\mathbb{R})$.

The most important example of a Fuchsian group is $\mathrm{PSL}_2(\mathbb{Z})$. Underlying this entire study and almost one and a half centuries of interactions between number theory and complex analysis is the remarkable fact that the quotient of $\mathbb{H}$ by certain discrete subgroups has the structure of a Riemann surface [Miy89, §1.8] and even an algebraic variety which, in important cases, is a moduli space [Mil12].

Definition 4.2.2 (Quotient of $\mathbb{H}$ by a group)**.** *If a group G acts on a set X, $G \setminus X$ has universe the collection of G-orbits of the action. π is the canonical map taking x to its orbit Gx. The prototypical example corresponds to $X = \mathbb{H}$.*

Definition 4.2.3. *The quotients $V = S(\mathbb{C})$ of $\mathbb{H}$ by a discrete group Γ that we consider are examples of* moduli spaces. *$V = \bigcup_{a \in C} V_a$ is the image of a map p from $\mathbb{H}$ that acts as a* uniformizer *for a family of varieties V_a. Namely for each $a, b \in \mathbb{H}$, $V_a \cong V_b$ iff for some $\gamma \in \Gamma$, $\gamma(a) = b$ iff $p(a) = p(b)$.*

We explored in Section 4.1 the ur-example of a moduli space, elliptic curves as uniformized by the j-function. The next definition relies on the fact that, while elements of $\mathrm{PSL}_2(\mathbb{R})$ fix $\mathbb{H}$ setwise, they also act on all of $\mathbb{C}$.

Definition 4.2.4 (Cusp)**.** *For a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$:*

1. *We say $c \in \mathbb{R} \cup \{\infty\}$ is a* cusp *of Γ if c is the unique fixed point of some $\gamma \in \Gamma$.*
2. *P_Γ is the set of cusps of Γ and $\mathbb{H}^* = \mathbb{H}^*_\Gamma = \mathbb{H} \cup P_\Gamma$.*

We relate some standard facts (see [Har14, p 15]). The first relies on the fact that while some of the quotients we study are not compact, they can be compactified by adding finitely many cusps from $\mathbb{R} \cup \{\infty\}$.

Fact 4.2.5. *For any discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$, the quotient $\Gamma\backslash\mathbb{H}^*_\Gamma$ is a compact Hausdorff space that can be given the structure of a Riemann surface. Therefore if Γ' is of finite index in Γ, the quotient $\Gamma' \setminus \mathbb{H}^*_\Gamma$ is a compact Riemann surface, and is therefore algebraic by the Riemann existence theorem. $\mathbb{H}^*_\Gamma$ is the compactification of the quasi-projective algebraic variety (so first order definable) $\mathbb{H}_\Gamma$, .*

For the purposes of this paper since the quasiprojective variety $\mathbb{H}_\Gamma = \Gamma \setminus \mathbb{H}$ determines the (classical) algebraic variety (set of solutions of a system of polynomial equations), $\mathbb{H}^*_\Gamma$ we work hereafter with $\mathbb{H}_\Gamma$. This is natural from a model-theoretic standpoint since (in this situation) there are only finitely many cusps and so the sets differ by only finitely many points.

Notation 4.2.6 *fixes the group G for the rest of § 4.* Setting the determinant as 1 and modding out the center guarantees the group action preserves both distance and orientation.

Notation 4.2.6. *Let $G = \mathrm{GL}_2^{ad}(\mathbb{Q})^+ =_{\mathrm{def}} PSL_2(\mathbb{Q})/Z(PSL_2(\mathbb{Q})) \approx PSL_2(\mathbb{Q})$ modulo its center. Γ varies over subgroups of G*

We now distinguish two kinds of points in $\mathbb{H}$: 'special' points and 'Hodge-generic' points. The equivalence of the following definition with the usual notion [DH17, Definition 2.2] for Shimura varieties is in [DH17, Theorem 2.3].

Definition 4.2.7 (Special points)**.** *Fix $\langle \mathbb{H}, S(\mathbb{C}), p\rangle$ with $S(\mathbb{C})$ biholomophic to $\Gamma \setminus \mathbb{H}$. A point $x \in \mathbb{H}$ is* special *if there is a $g \in G$ whose unique fixed point is x.*

We omit the definition of a Hodge generic point arising in algebra, as it does not enter our discussion; we use only the equivalent characterization [DH17, Prop 2.5] given in Fact 4.2.8.1) and the dichotomy in 2) noted just after that proposition. It is worth mentioning that for a point the fact of being "special" or "Hodge generic" does not depend on the choice of the group Γ; furthermore, these two notions are preserved by the action of $G = GL_2^{\mathrm{ad}}(\mathbb{Q})^+$.

Fact 4.2.8. *Special and Hodge generic points [DH17, Proposition 2.5]*

(1) If x is Hodge generic the only $g \in G$ that fixes x is the identity.

(2) Every point in $\mathbb{H}$ is either Hodge generic or special.

Although we are studying the categoricity of the universal cover of a specific modular curve (e.g. the image of the j-function, $\Gamma\backslash\mathbb{H}$), other modular curves naturally arise in the analysis. The study of families of such curves is expounded in [Shi71, §6, 7]. A key tool to give a uniform treatment to a family is the existence of a common commensurator of the generating Fuchsian groups. In fact, the members of the family are interalgebraic and the entire family (indexed by the Γ_N) is studied in [DZ22a].

Definition 4.2.9. *1. The groups Γ_N (N a fixed integer) are given by*

$$\Gamma_N = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : \ b \equiv c \equiv 0, a \equiv d \equiv 1 \bmod N \right\}.$$

Note that each Γ_N has finite index in Γ and if $N|M$ then $\Gamma_M \subseteq \Gamma_N$.

2. *Two subgroups Γ and Γ' of a group H are said to be* commensurable *if $\Gamma \cap \Gamma'$ is of finite index in both of them.*
3. *A* congruence subgroup *is a subgroup Γ' of Γ such that some Γ_N is a finite index subgroup of Γ'.*
4. *The* commensurator $\mathrm{comm}(\Gamma)$ *of a subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ is*

$$\{\delta \in \mathrm{PSL}_2(\mathbb{R}) \colon \delta\Gamma\delta^{-1} \text{ is commensurable with } \Gamma\}.$$

We rely on the following standard fact.

Lemma 4.2.10. *The group $G = \mathrm{GL}_2^{ad}(\mathbb{Q})^+$ (Notation 4.2.6) is the commensurator of any congruence subgroup Γ of $SL_2(Z)$.*

Because the functions $g \in G$ are in the formal vocabulary, we employ congruence subgroups $\Gamma_{\mathbf{g}}$ from Notation 4.2.11 rather than the Γ_N. The $Z_{\mathbf{g}}$ defined in Notation 4.2.11 play a central role both in the quantifier elimination and via an inverse limit in Section 4.4.

Notation 4.2.11. *With G as fixed in Notation 4.2.6, as each of the congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ act on $\mathbb{H}$ we can define for any finite sequence of the form $\mathbf{g} = \langle e, g_2, \dots, g_n\rangle$ from G (by convention, $g_1 = e$),*

1. $\Gamma_{\mathbf{g}} = \Gamma \cap g_2^{-1}\Gamma g_2 \ldots \cap g_n^{-1}\Gamma g_n$.
2. *Let $p : \mathbb{H} \to S(\mathbb{C})$.*
   (a) $Z_{\mathbf{g}}$ is defined as $\{(p(x), p(g_2x), \dots, p(g_nx)) \in S(\mathbb{C})^n : x \in \mathbb{H}\}$.

(b) *Let* $p_{\mathbf{g}} : \mathbb{H} \to Z_{\mathbf{g}} \subseteq S(\mathbb{C})^n$ *be defined by*

$$x \mapsto p(\mathbf{g}x) = \langle p(x), p(g_1(x)), \ldots, p(g_n(x)) \rangle.$$

(c) *Let* $[\phi_{\mathbf{g}}]$ *be the map from* $\mathbb{H}_{\mathbf{g}}$ *onto* $Z_{\mathbf{g}}$ *(Lemma 4.2.12) given by* $[\phi_{\mathbf{g}}]x_{\Gamma_{\mathbf{g}}} = p_{\mathbf{g}}(x)$.

3. *Let* $\mathbb{H}_{\mathbf{g}}$ *denote* $\Gamma_{\mathbf{g}} \setminus \mathbb{H}$.

All the previous are well-defined by our choice of p and Γ.

The following lemma [Ete22, 3.31] is central to Section 4.4.2. Its proof uses Shimura theory very heavily.

Lemma 4.2.12. *The map* $[\phi_{\mathbf{g}}]$ *is bijective on the Hodge generic points and the image* $Z_{\mathbf{g}}$ *is a* **variety** *contained in* $\mathcal{S}^n(\mathbb{C})$, $n = \lg(\mathbf{g})$. *Moreover, [Ete22, p 17], for all* $\mathbf{g}$, $Z_{\mathbf{g}}$ *is defined over the maximal Abelian extension L of the field of definition, E, of S.*

Remark 4.2.13. From the model theoretic standpoint, it makes no sense to say the $[\phi_{\mathbf{g}}]$ are definable since their domains $\mathbb{H}_{\mathbf{g}}$ are not. While the maps $[\phi_{\mathbf{g}}]$ are bijective on Hodge generic points, they may identify special points.

4.3 Quantifier Elimination in Modular and Shimura Curves

We now lay out the vocabulary and first order theory for studying modular curves. The mathematical input is a Fuchsian group Γ acting on hyperbolic space $\mathbb{H}$ and the image curve $S(\mathbb{C}) = \Gamma \setminus \mathbb{H}^*_\Gamma$ (Definition 4.2.4) with a standard model $\mathbf{p} = \langle \mathbb{H}, S, p \rangle$. The structure of a discrete group is unwieldy from a traditional model theoretic standpoint because its first order theory is unstable and undecidable. Just as modules are usually studied in model theory by adding unary function symbols f_r for the elements of the ring, in order to represent the action of G on $\mathbb{H}$, we add symbols f_g for $g \in G$ as unary functions that act on $\mathbb{H}$. We thus use a two-sorted presentation of our structures: a sort for the domain, a sort for the target, and a map p connecting them.

Remark 4.3.1 (Sorts)**.** A two-sorted structure interprets two sort symbols and additional relation and function symbols with the understanding that each such relation/function either is restricted to one of the predicates or explicitly connects them.

Notation 4.3.2 (The formal vocabulary τ)**.** The two-sorted vocabulary τ consists of the sorts (unary predicate symbols) D (the covering sort), S the target sort, and a function q mapping D onto the sort S.

We write τ_G for the vocabulary of the first sort with $G = G^{ad}(\mathbb{Q}^+)$. The second $\tau_F = \mathcal{R}$ where $\mathcal{R}$ is the set of formulas in $\{+, -, 0, 1, \times\}$ specified in Definition 4.3.3. τ is $\tau_G \cup \tau_F \cup \{p\}$. There are *constant symbols* for each element of the field $E^{ab}(\Sigma)$ defined in Notation 4.3.3. We use f_g to name the functions acting on D, but often write the shorter $g(x)$ or gx instead of $f_g(x)$.

The following notation is essential to understand the Axioms 4.3.5. Note in the prototype q is replaced by the known covering map p.

Notation 4.3.3. The *standard model* for a *modular curve* determined by a Fuchsian group $\Gamma \subseteq G = G^{ad}(\mathbb{Q}^+)$ will consist of a τ-structure $\mathbf{p} = \langle \mathbb{H}, S, p \rangle$ with the domain $\mathbb{H}$, the variety $S(F)$ over the algebraically closed field F defined by $\Gamma \setminus \mathbb{H}$, and $\mathcal{R}$ the set of all Zariski closed relations on $S(F)^n$ (for all n) with constants from a field $E^{ab}(\Sigma)$ that are true in F. E^{ab} is the maximal abelian extension of the defining (reflex) field E of S. $E^{ab}(\Sigma)$ is the extension of E^{ab} (F_0 in [Ete22, p 19]) obtained by adding the coordinates of the ($\leq \aleph_0$) special points, and closing to a field.

Notation 4.3.4. For a structure $\mathbf{p}$, we write $\mathrm{Th}(\mathbf{p})$ for the complete first order theory of all sentences true in $\mathbf{p}$ and $T(\mathbf{p})$ for the specified set of axioms true of $\mathbf{p}$. Clearly, $T(\mathbf{p}) \subseteq Th(\mathbf{p})$.

We must distinguish $\mathrm{Th}(\mathbf{p})$ from its subset $T(\mathbf{p})$ until we prove $T(\mathbf{p})$ is a complete axiomatization of $\mathrm{Th}(\mathbf{p})$.

Definition 4.3.5 (First Order Axioms)**.** $T(\mathbf{p})$ *is the following collection of first order sentences that are to hold in a structure* $\langle D, S(F), q \rangle$.

1. *Each sentence in* $\mathrm{Th}(\langle \mathbb{H}, \{f_g : g \in G\} \rangle)$. *These include 'Special Point axioms'* SP_g*: For each* $g \in G$ *that fixes a unique point in* D

$$\forall x, y \in D[(g(x) = x \wedge g(y) = y) \Rightarrow x = y]$$

2. $\mathrm{Th}(S(\mathbb{C}), \mathcal{R})$ *($\mathcal{R}$ from Definition 4.3.2.)*
3. *The covering map; for each* $\mathbf{g} \in G^m$ *and all* $m < \omega$*:*
   (a) $Mod^1_{\mathbf{g}}$,
   $$\forall x \in D\ (q(g_1(x), \dots q(g_m(x)) \in Z_{\mathbf{g}})$$
   (b) $Mod^2_{\mathbf{g}}$*:*
   $$\forall z \in Z_{\mathbf{g}} \exists x \in D(q(g_1(x)), \dots q(g_m(x)) = z)$$
   (c)
   $$\mathbf{MOD} = \{Mod^1_{\overline{g}} \wedge Mod^2_{\overline{g}} : \overline{g} \in G^m, m < \omega\}$$

Note that $\mathbf{MOD}$ is a countable collection of first order sentences.

Notation 4.3.6. By the choice of $E^{ab}(\Sigma)$, special points belong to $\mathrm{dcl}\,(\emptyset)$. Therefore, we can name each one of them by d_g, where $g \in G$ fixes d_g. Any g that fixes a point is in $G - \mathrm{SL}_2(\mathbb{Z})$ [Ete22, Lemma 3.18]. There will be distinct g_1, g_2 that fix the same point (e.g. if $g_2 = g_1^2$). If so, $T(\mathbf{p}) \vdash d_{g_1} = d_{g_2}$ The theory of (D, G) contains the uniqueness axiom (Definition 4.3.5.1) that entails $g(d_g) = d_g$.

The cover sort is a set with unary functions. Both its theory (since the universe is a union of orbits) and that of the field sort (since algebraically closed) are strongly minimal and quantifier eliminable.

Definition 4.3.7. We say two structures M and N are **qf**-back and forth equivalent if the system I of partial isomorphisms of M and N between isomorphic *finitely generated substructures* satisfies the back and forth condition: For each $f \in I$ and each $m \in M - \operatorname{dom} f$, there exists an $n \in N$ such that $f \cup \{\langle m, n \rangle\} \in I$, and symmetrically, for each $n \in N - \operatorname{im} f$, there exists $m \in M$ such that $f \cup \{\langle m, n \rangle\} \in I$. In this situation $\operatorname{dom} f$ is definably close.

Notation 4.3.8. We write $\mathbf{g}(x)$ for $(g_1(x), \ldots g_n(x))$ where $\mathbf{g}$ has length n and begins with e. And then $\mathbf{g}(\mathbf{x})$ denotes the sequence of length nm obtained when $\mathbf{g}$ is applied to each element of a sequence $\mathbf{x} \in (D)^m$. When convenient we write gx or $\mathbf{g}x$ for the action, omitting the parentheses.

We now sketch the proof of Theorem 4.3.13 that $T(\mathbf{p})$ axiomatizes a complete, quantifier eliminable τ-theory.

Definition 4.3.9 (The back and forth)**.** Fix two models $\mathbf{q} = \langle D, S(F), q \rangle$ and $\mathbf{q}' = \langle D', S(F'), q' \rangle$ of $T(\mathbf{p})$. We define the **qf**-back-and-forth system I of substructures of $\mathbf{q}$ and $\mathbf{q}'$ For each $f \in I$, $\operatorname{dom} f$ and $\operatorname{rg}\ f$ are each finitely generated over $E^{ab}(\Sigma)$. A typical member f of the system for $\mathbf{q}$ has $\operatorname{dom} f = U = U_D \cup U_S$. Since U is finitely generated, U_D consists of the G-orbits of a finite number of $x \in D$; U_S is $S(L_U)$ where L_U is the field generated by $E^{ab}(\Sigma)$ (since the elements of $E^{ab}(\Sigma)$ elements are named), the coordinates of the $q(x)$ for $x \in U_D$ and finitely many additional points of $F \cap U$. Note that the additional points determine finitely many new field elements since q is constant on each orbit, so the field remains finitely generated. Define a similar subsystem for $\mathbf{q}'$, labeling by putting primes on corresponding objects. By Lemma 4.2.8 every point of D is either special and so named in the vocabulary (Remark 4.3.6), or Hodge generic. Thus we can ignore the special points in building the back and forth system.

Suppose f is an isomorphism between $U \subseteq \mathbf{q}$ and $U' \subseteq \mathbf{q}'$. Then f restricts to a G-equivariant (elements in the same orbit have the same image) injection of U_D into $U_{D'}$ and an embedding of $S(L_U)$ into $S(F')$ induced by an embedding σ of L into $S(F')$, that fixes $E^{ab}(\Sigma)$.

Note that the following claim is for arbitrary finite sequences $\mathbf{g}$, but only singleton x. The type r_d of an infinite sequence (here represented by an infinite tuple of variables $\mathbf{v}$) includes the types of $\mathbf{g}x$ for any finite $\mathbf{g}$.

The main consequence of the following claim is that we may reduce types of points in the domains sort to quantifier-free types of their images in the field sort.

Claim 4.3.10. *[DH17, Prop 3.3] If $d \in D - U_D$ is Hodge generic:*

$$r_d(\mathbf{v}) \models \mathrm{tp}_{qf}(d/U),$$

where $r_d(\mathbf{v}) = \bigcup_{\mathbf{g}\in G} \mathrm{tp}_{qf}(q(\mathbf{g}(d))/U) = \mathrm{tp}_{qf}(\langle q(g(d)) : g \in G\rangle/U)$.

Proof. We show that there is a unique quantifier-free type over U of an element of D that restricts to r_d. The consistent non-trivial types in τ_G are i) $\{x \neq f : f \in U_D\}$ and ii) $\{x \neq gx\}$ for any non-identity $g \in G$. The first is captured by $(q(x), q(f)) \notin Z_{e,e}$ for each $f \in U_D$ and the second by $(q(x), q(x)) \notin Z_{e,g}$ if $g \notin \Gamma$ and these are both in $r(\mathbf{v})$.

Suppose $\mathbf{h} \in S(M)^\omega$ (for a saturated $M \models T(\mathbf{p})$ containing U) realizes $r_d(\mathbf{v})$ and $\mathbf{h}$ with $d' \in D(M)$ satisfy $\mathbf{h} = \langle q(g(d')) : g \in G\rangle$. By the previous paragraph $d' \notin U_D$. So d' realizes $\mathrm{tp}_{qf}(d/U)$ as required. □

Notation 4.3.11. *For a type $r(v)$ over a set A and an isomorphism f from A to B, $f(r)$ is the set of B-formulas $\phi(v, f(\mathbf{a}))$ with $\phi(v, \mathbf{a}) \in r$.*

Claim 4.3.12. *[DH17, Prop 3.4] Fix $\mathbf{g}$. If $x \in U_D$, there is an $x' \in U_{D'}$ such that $q(\mathbf{g}(x')) \in S(F')^m$ realizes $f(\mathrm{tp}_{qf}(q(\mathbf{g}(x))/L_U))$.*

Proof. We write $Z^{\mathbf{q}}_{\mathbf{g}}$ for the points in $S(F)$ satisfying (the formula defining) $Z_{\mathbf{g}}$. Using Notation 4.3.11, Claim 4.3.10 implies that the smallest algebraic subvariety $W^{\mathbf{q}}_{\mathbf{g}}$ of $S(F)^n$ that is defined over L_U and contains $q(\mathbf{g}(\mathbf{x})) \in S(F)^n$ determines $\mathrm{tp}_{qf}(\mathbf{g}(\mathbf{x}))/L_U)$. Since $Mod^1_{\mathbf{g}}$ is true in $\mathbf{q}$, $W^{\mathbf{q}}_{\mathbf{g}} \subseteq Z^{\mathbf{q}}_{\mathbf{g}}$. But since (by Lemma 4.2.12) $Z^{\mathbf{q}}_{\mathbf{g}}$ is fixed setwise by σ (the map described after Definition 4.3.9) - being defined over $E^{ab}(\Sigma)$, we have that $Z^{\mathbf{q}'}_{\mathbf{g}} = Z^{\mathbf{q}}_{\mathbf{g}}$, and therefore $W^{\mathbf{q}'}_{\mathbf{g}} \subseteq Z^{\mathbf{q}'}_{\mathbf{g}}$. Now applying $Mod^2_{\mathbf{g}}$ in $\mathbf{q}'$, we find the required x'. □

Having proved Claim 4.3.12, we can finish the argument. We need one more crucial piece for the 'forth'. What if $x \in D - U_D$? For this, we need $\mathbf{q}'$ to be ω-saturated (realize all types over finite sets).

Theorem 4.3.13. *Suppose that $\mathbf{q}$ and $\mathbf{q}'$ are ω-saturated. Then the qf-system described in Remark 4.3.9 is a back and forth; hence, $T(\mathbf{p})$ admits elimination of quantifiers and is complete.*

Proof. Suppose f is an isomorphism between $U \subseteq \mathbf{q}$ and $U' \subseteq \mathbf{q}'$. Then f restricts to a G-equivariant injection of U_D into $U_{D'}$ and an embedding of $S(L_U)$ into $S(F')$ induced by an embedding σ of L_u into $S(F')$, that fixes $E^{ab}(\Sigma)$.

For $x \in \mathbf{q} - U$, we must find $x' \in U'$ so that $f \cup (x, x')$ generates an isomorphism between the structures generated by $U \cup \{x\}$ and $U' \cup \{x'\}$. If $x \in S$, $x = q(\check{x})$ for some $\check{x} \in D$ so we restrict to that case. If $x \in U_D$, x' exists as U'_D is closed under action by G. Since the coordinates of special points are in $E^{ab}(\Sigma)$, whose points are all named, for a special point x, x' must equal x.

The difficult case is when $x \in (D - U_D)$ is Hodge generic. But we noted in Claim 4.3.10 that it suffices to simultaneously realize all types $\mathrm{tp}_{qf}((q(g_1x), \ldots q(g_nx))/U)$ for all $\mathbf{g}$ (of arbitrary length). A slight variant on the argument for Claim 4.3.12 still holds if for fixed x, we replace a single $\mathbf{g}$ by an arbitrary finite set of $\mathbf{g}$. By compactness, the entire type is consistent and so satisfied in the ω-saturated $\mathbf{q}'$. There is one final step. By induction we have to choose x' for a sequence $\mathbf{x}, \mathbf{y}, x$ where $\mathbf{x} \in U_D$ and $\mathbf{y} \in U_S^k$ for some k. But what if $x \in U_S$? By Claim 4.3.10, $\mathrm{tp}_{qf}(\mathbf{x}, \mathbf{y})$ is determined by $\mathrm{tp}_{qf}(\mathbf{g}(\mathbf{x}), \mathbf{y})$ (in the field sort). That we can choose of $x' \in U'_S$ to satisfy $f(\mathrm{tp}_{qf}(\mathbf{g}(\mathbf{x}), \mathbf{y}))$ is now immediate by ω-saturation and quantifier elimination in the field-sort.

By Karp's theorem [Bar73, Theorem 3], the existence of the back and forth implies all ω-saturated models of $T(\mathbf{p})$ are $L_{\omega_1,\omega}$ (indeed, $L_{\infty,\omega}$) elementarily equivalent. Every model has an ω-saturated elementary extension, so $T(\mathbf{p})$ is complete. $\square$

4.4 Galois Representations and finite index conditions

In this section we begin by considering the action of discrete and Galois groups on the domain and field sorts. Then we unite these approaches by defining a Galois representation. We then state the key to establishing categoricity, a consequence of Serre's open mapping theorem.

4.4.1 Two views: domain and field sort

We explore the following diagram which links the domain sort (via the quotient) with the field sort.

$$\begin{array}{ccc} \mathbb{H}_{\overline{h}} \approx \Gamma_{\overline{h}} \setminus \mathbb{H} & \xrightarrow{[\phi_{\overline{h}}]} & Z_{\overline{h}} \\ {\scriptstyle \mathrm{id}_{\mathbb{H}_{\overline{g}}}} \downarrow & & \downarrow {\scriptstyle \psi_{\overline{h},\overline{g}}} \\ \mathbb{H}_{\overline{g}} \approx \Gamma_{\overline{g}} \setminus \mathbb{H} & \xrightarrow{[\phi_{\overline{g}}]} & Z_{\overline{g}} \end{array}$$

Convention 4.4.1. $\mathbf{g} = \langle e, g_1 \ldots g_{n-1}\rangle$ *has length* n. *We restrict to* $\mathbf{g}$ *with* $\Gamma_{\mathbf{g}} \trianglelefteq \Gamma$ *(normal subgroup). Recall* $Z_{\mathbf{g}} \subseteq S(\mathbb{C})^{\mathrm{lg}(\mathbf{g})}$.

We have two views of 'essentially' the same map. The first moves to a quotient on the domain side which is not τ-definable; the second 'names' the range of the first in the target side. We begin with *quotient data* but with manifestations in both the domain and target.

Domain/Quotient data: The first view motivates id for identity.

Definition 4.4.2. *Let* $\mathbf{g} \subseteq \mathbf{h}$*. Define* $\mathrm{id}_{\mathbf{hg}} : \mathbb{H}_{\mathbf{h}} \to \mathbb{H}_{\mathbf{g}}$ *by* $[x]_{\Gamma_{\mathbf{h}}} \mapsto [x]_{\Gamma_{\mathbf{g}}}$*.*

The normality hypothesis implies that $\Gamma_{\mathbf{g}}/\Gamma_{\mathbf{h}}$ acts on $\mathbb{H}_{\mathbf{g}}$: for $\lambda \in \Gamma_{\mathbf{g}}$, $\lambda[x]_{\Gamma_{\mathbf{g}}} := [\lambda x]_{\Gamma_{\mathbf{g}}}$, so the representatives λ_i of the cosets of $\Gamma_{\mathbf{g}}/\Gamma_{\mathbf{h}}$ index the equivalence classes; thus the action is transitive.

Field data: We define the right hand column of the diagram.

Definition 4.4.3. *1. For* $\mathbf{g} \subseteq \mathbf{h}$*,* $\lg(\mathbf{g}) = n$*,* $\lg(\mathbf{h}) = m$*,* $\psi_{\mathbf{h},\mathbf{g}}$*, denotes the restriction of the natural projection from* $S(\mathbb{C})^m$ *onto* $S(\mathbb{C})^m$ *to a map from* $Z_{\mathbf{h}} \subseteq S(\mathbb{C})^m$ *onto* $Z_{\mathbf{g}} \subseteq S(\mathbb{C})^n$*.*

2. *Choose* $z \in Z_{\mathbf{g}}$ *and let* $L = L_z$ *be a finitely generated extension of the defining field for* S *such that* z *is defined over* L*. Write* $\overline{\mathrm{L}}$ *for* $\mathrm{acl}(L)$*.*

3. *Now,* $\mathrm{Aut}(\mathbb{C}/\mathrm{L})$ *acts on the fiber of* $\psi_{\mathbf{h},\mathbf{g}}$ *over* z*, by its action on the coordinates of* z*; as it would for any definable finite-to-one map from* $Z_{\mathbf{h}}^m \to Z_{\mathbf{g}}^n$*.*

To connect the two sides, conjugating by $[\phi_{\mathbf{h}}]$, $\mathrm{Aut}(\overline{\mathrm{L}}/\mathrm{L})$ acts on $\mathrm{id}_{\mathbf{hg}}^{-1}(\mathrm{z})$.

Lemma 4.4.4. *[Ete22, §3.5 p. 18]* $\mathrm{Aut}(\mathbb{C}/\mathrm{L})$ *acts on the fiber of* $\psi_{\mathbf{h},\mathbf{g}}$ *over* z*, (and so via* $[\phi_{\mathbf{h}}]$ *on* $\mathrm{id}_{\mathbf{hg}}^{-1}(\mathrm{z})$*). This action commutes with the action of the free and transitive (simply transitive) action of* $\Gamma_{\mathbf{g}}/\Gamma_{\mathbf{h}}$ *on the fibers of* $\mathrm{id}_{\mathbf{h},\mathbf{g}}$*. Thus we have a homomorphism (Galois representation)* $\rho^z_{\mathbf{g},\mathbf{h}}$ *from* $\mathrm{Aut}(\overline{\mathrm{L}}/\mathrm{L})$ *into* $\Gamma_{\mathbf{g}}/\Gamma_{\mathbf{h}}$*.*

4.4.2 Galois Representation

While the notion of a representation of a group A frequently refers to linear representations, a homomorphism of A into a matrix group B, here we will discuss specific examples of a more general notion: *a representation of* A *is a homomorphism of* A *into a group* B*. This is a* Galois representation *if* A *is the Galois group of one field over another.* In Section 4.4.1, we gave Galois representations of $\mathrm{Aut}(\overline{\mathrm{L}}/\mathrm{L})$ into $\Gamma_{\mathbf{g}}/\Gamma_{\mathbf{h}}$. In order to understand how to combine the actions of the $\Gamma_{\mathbf{g}}/\Gamma_{\mathbf{h}}$ as $\mathbf{g}, \mathbf{h}$ vary, we need the notion of inverse limit.

Definition 4.4.5 (Inverse Limit)**.** *Given a directed set* $(I, \leq)$ *an* inverse system *on* I *is a family of structures* $\langle A_i : i \in I\rangle$*, and for* $i < j$*, maps* f_{ij} *from* A_j *to* A_i *such that* $i < j < k$ *implies* $f_{ij} \circ f_{jk} = f_{ik}$*.*

An inverse limit *of this inverse system is an object* $\hat{A} = \varprojlim A_i$ *and a family of morphisms* $g_i : \hat{A} \to A_i$ *such that (1) for all* $i < j$ *in* I*,* $f_{ij} \circ g_j = g_i$ *and (2) given any* A' *and family* ${g'}_i$ *satisfying (1) there is a unique morphism* $h : \hat{A} \to A'$ *such that for all* $i \in I, {g'}_i = g_i \circ h$*.*

Definition 4.4.6. Galois Representations of Inverse Limits *We work with a modular curve* $S(\mathbb{C}) = \Gamma \setminus \mathbb{H}$ *which is defined over* $E^{ab}(\Sigma)$*. (Notation 4.3.3). Since each*

$\Gamma\mathbf{g} \subseteq \Gamma$, $\rho^z_{\mathbf{g},\mathbf{h}} : \mathrm{Aut}(\overline{\mathrm{L}}/\mathrm{L}) \to \Gamma$ *and by taking an inverse limit of the representations* $\rho^z_{\mathbf{g},\mathbf{h}}$*, we obtain:*

$$\rho^z \colon \mathrm{Gal}(\overline{L}/L) \to \overline{\Gamma}$$

where $\overline{\Gamma} = \varprojlim_{\mathbf{h}} \Gamma/\Gamma_{\mathbf{h}}$. *The* $\mathbf{h}$ *range over all finite sequences as Convention 4.4.1. See Definition 4.4.5 and [Ete22, §3.6 p 17].*

For any groups $H_1 \leq H_2$ that act on a set X the H_1-orbits of X partition the H_2-orbits. So if $[H_2 : H_1]$ is finite and H_2 is infinite, the obits will have the same cardinality and the smaller $[H_2 : H_1]$ is, the closer we are to an isomorphism.

Now, we can state the first of two crucial sufficient conditions for categoricity.

Definition 4.4.7. First Finite Index Condition (FIC1) *The* first finite index condition *is satisfied by a modular curve* $p\colon \mathbb{H} \to S(\mathbb{C})$ *if:*

For any non-special points $x_1, \ldots x_m \in \mathbb{H}$ *in distinct* G*-orbits (Definitions 4.4.2, 4.4.3) and for any field* L *containing the field over* $E^{ab}(\Sigma)$ *along with the coordinates of the* $p(x_i)$*, the image of the induced homomorphism* $\rho : \mathrm{Gal}(\overline{L}/L) \to \overline{\Gamma}^m$ *has finite index in* $\overline{\Gamma}^m$.

Recall from Lemma 4.3.10 that

$$r_d(\mathbf{v}) \models \mathrm{tp}_{qf}(d/U).$$

where $r_d(\mathbf{v}) = \bigcup_{\mathbf{g}\in G} \mathrm{tp}_{qf}(q(\mathbf{g}(d))/U) = \mathrm{tp}_{qf}(\langle q(gd) : g \in G\rangle/U)$. The argument for Lemma 4.3.10 began with the observation that $r_d(\mathbf{v})$ implied, in particular, that $d \notin D_U$, so d is an independent Hodge generic. We will deduce from Lemma 4.4.8 that (under FIC1) only finitely many tuples $\mathbf{g}$ from r_d are really needed.

Lemma 4.4.8. *Assume FIC1. Then, for each* z*, for some* $\hat{\mathbf{g}}$*, the map*

$$\rho_z : \mathrm{Aut}(\overline{\mathrm{L}}/\mathrm{L}_{\hat{\mathbf{g}}}) \mapsto \overline{\Gamma}^{\mathrm{m}}_{\hat{\mathbf{g}}} = \varprojlim_{\mathbf{h}\supseteq\hat{\mathbf{g}}} (\Gamma_{\hat{\mathbf{g}}}/\Gamma_{\mathbf{h}})^{\mathrm{m}}$$

is surjective.

Proof. Let $I = \mathrm{im}(\rho_z)$ and let $k = [\overline{\Gamma} : I]$. Suppose not. Choose $\hat{\mathbf{g}}$ with $\mathbf{g} \subseteq \hat{\mathbf{g}}$ such that $[\Gamma_{\mathbf{g}} : \Gamma_{\hat{\mathbf{g}}}] = k$. Thus, for any $\mathbf{h} \supseteq \hat{\mathbf{g}}$, ρ_z must be onto $\Gamma_{\hat{\mathbf{g}}}/\Gamma_{\mathbf{h}}$. For, if not, there is an $\eta \in \Gamma_{\hat{\mathbf{g}}}/\Gamma_{\mathbf{h}}$ and that is not in I; it must be in a new coset of I in $\overline{\Gamma}$, contrary to the choice of $\hat{\mathbf{g}}$. □

Corollary 4.4.9. *(Under FIC1) For* $d \in D - U$*,*

$$\mathrm{tp}_{qf}(q(\hat{\mathbf{g}}(d))/U) \models r_d(\mathbf{v}) \models \mathrm{tp}_{qf}(d/U).$$

Proof. The second implication is Lemma 4.3.10. For the first, choose any $\mathbf{h} \supseteq \hat{\mathbf{g}}(d)$ and let $m = \lg(\hat{\mathbf{g}})$, $r = \lg(\mathbf{h})$. Let $\mathcal{F} \subseteq Z^r_{\mathbf{h}}$ be the fiber over $\hat{\mathbf{g}}(d') \in Z^m_{\hat{\mathbf{g}}}$ of the finite-to-one map $\psi_{\mathbf{h}\hat{\mathbf{g}}} : Z^r_{\mathbf{h}} \to Z^m_{\hat{\mathbf{g}}}$. Similarly, $\mathrm{tp}_{qf}(\mathbf{h}(d)/L_U)$ is determined by the $\mathrm{Aut}(\mathbb{C}/\mathrm{L_U})$-orbit $\mathcal{G} \subseteq \mathcal{F}$ containing $\mathbf{h}(d)$. Then, $\mathrm{tp}_{qf}(\mathbf{h}(\mathbf{x})/L_U)$ is determined by the $\mathrm{Aut}(\mathbb{C}/\mathrm{L_U})$-orbit $\mathcal{G} \subseteq \mathcal{F}$ containing $\mathbf{h}(\mathbf{x})$. But $\mathcal{G} = \mathcal{F}$, since ρ_z induces a homomorphism from $\mathrm{Aut}(\mathbb{C}/\mathrm{L_U})$ *onto* $\Gamma_{\hat{\mathbf{g}}}/\Gamma_{\mathbf{h}}$ and $\Gamma_{\hat{\mathbf{g}}}/\Gamma_{\mathbf{h}}$ acts transitively on the fiber. Since this holds for any such $\mathbf{h}$, we finish. $\square$

We turn now to the infinitary axioms that are needed to obtain categoricity.

Notation 4.4.10 (Infinitary Axioms)**.** *1. Φ_∞ is the $L_{\omega_1,\omega}$ sentence asserting that for (D, S, q) both the dimension of the field bi-interpretable with S and of the strongly minimal structure $\langle D, \{f_g : g \in \Gamma\}\rangle$ are infinite.*

2. SF (standard fibers) denotes the $L_{\omega_1,\omega}$-axiom:

$$(\forall x \forall y \in D)(q(x) = q(y) \to \bigvee_{g \in \Gamma} x = f_g(y)).$$

3. $T^\infty(\mathbf{p})$ denotes $\mathrm{Th}(\mathbf{p}) \cup \{\Phi_\infty\}$ and

4. $T^\infty_{SF}(\mathbf{p})$ denotes $\mathrm{Th}(\mathbf{p}) \cup \{SF\} \cup \{\Phi_\infty\}$.

Definition 4.4.11. *For $\langle D, S(F), q\rangle \models T^\infty_{SF}(\mathbf{p})$ and $X \subset D \cup S(F)$,*

$$\mathrm{cl}(X) = q^{-1}(\mathrm{acl}(q(X)))$$

where acl *is the field algebraic closure in F.*

An essential consequence of the standard fibers axiom is that Definition 4.4.11 defines an almost quasiminimal closure relation satisfying the countable closure condition from Definition 3.2.4. This closure dimension restricts on the separate sorts to the dimension of the constituent strongly minimal sets that is expressed in Φ_∞. This accomplishes the aim of an ($L_{\omega_1,\omega}$-complete so $\aleph_0$-categorical) $L_{\omega_1,\omega}$ theory with arbitrarily large models.

A class K of models has $\aleph_0$-homogeneity over $\emptyset$ (Definition 3.2.5) (the precise statement is from [Ete22, p 4]) if the models of K are pairwise **qf**-back and forth equivalent (Definition 4.3.7).

Theorem 4.4.12. *[DH17, Theorem 4.11] If the standard model $\mathbf{p}$ of a modular curve satisfies FIC1, then the class of models of $T^\infty_{SF}(\mathbf{p})$ is $\aleph_0$-homogenous over $\emptyset$. In particular, by Karp [Kar64, Bar73], all models of $T^\infty_{SF}(\mathbf{p})$ are back and forth equivalent and so satisfy the same sentences of $L_{\omega_1,\omega}$.*

Proof. Our task is to replace the ω-saturation hypothesis from Lemma 4.3.13 by adding the infinitary axioms and the condition FIC1. As in the proof of theorem 4.3.13 we need only worry about Hodge generic points. Suppose we have a partial function f from $\mathbf{q}$ to $\mathbf{q}'$ with domain and range U and U' as in Lemma 4.3.13 between models $\mathbf{q}$ and $\mathbf{q}'$ of $T^{\infty}_{SF}(\mathbf{p})$. Proceed as in the proof of the second paragraph of Lemma 4.3.13. We vary the argument for the 'difficult case' from the 3rd paragraph. Choose $\hat{\mathbf{g}}$ by Lemma 4.4.8. Taking $\hat{\mathbf{g}}$ for the $\mathbf{g}$ in Lemma 4.3.12, for $x \in U_D$, there is an $x' \in U_{D'}$ such that (*) $q(\hat{\mathbf{g}}(x')) \in S(F')^m$ realizes $f(\mathrm{tp}_{qf}(q(\hat{\mathbf{g}}(x))/L_U))$. We want to show that the same choice x' satisfies (*) for every $\mathbf{h} \supseteq \hat{\mathbf{g}}$. This is immediate from Lemma 4.4.9. The argument is completed by induction as in the 'final step' of the proof of Lemma 4.3.13.

□

Remark 4.4.13 (FIC2)**.** Like FIC1, FIC2 is a finite index condition on Galois representations into inverse limits. Now, however there are independence conditions over the ground field. [DH17, Condition 4.8] provides sufficient conditions so that a minor modification of the proof of Theorem 4.4.12, shows FIC2 implies homogeneity over models; pairs of models are back and forth equivalent over a countable submodel. *This is the first place in the argument where types over countable algebraically closed fields rather than the empty set (i.e. a fixed countable field) are encountered.* Combining this result with Theorem 4.4.12, the homogeneity conditions are now stronger than those defining quasiminimal excellence in [BHH$^+$14]. Thus, we apply that paper and obtain:

Theorem 4.4.14. *For any modular curve interpreted as a standard model* $\mathbf{p}$ *(Definition 4.3.3) for* $T^{\infty}(\mathbf{p})$*,* $T^{\infty}(\mathbf{p})$ *is almost quasiminimal excellent and so categorical in every infinite power.*

Proof. We need only that FIC1 and FIC2 hold for all modular curves. This is proved in [DH17, §5], where the proof for FIC1 relies heavily on [Ser72, §6] and FIC2 on [Rib75]. □

With further effort they extend this result to Shimura curves.

Remark 4.4.15. Keisler's theorem [Kei70, Corollary 5.10] and work of Shelah [Bal09, §7] show that an $\aleph_1$-categorical sentence ϕ of $L_{\omega_1,\omega}$ not only has only countably many types in any countable fragment of $L_{\omega_1,\omega}$ containing ϕ (Keisler) but has a completion[5] (Shelah). Equivalently, the completion must specify the isomorphism type of the countable model. The only such completion consistent with having an uncountable model is adding Φ_∞.

[5]That is, a sentence ϕ^* that implies ϕ and decides every $L_{\omega_1,\omega}$-sentence.

We have used FIC1 to prove categoricity in all powers. In fact, $\aleph_1$-categoricity implies FIC1. For this, [DH17, Ete22] argue that the weaker hypothesis of having just countably many types over the empty set in the theory T^{∞}_{SF} implies FIC1. If FIC1 holds, for some z, by Lemma 4.4.8, for every $\mathbf{g}$, there is $\mathbf{h} \supseteq \mathbf{g}$ with a $\Gamma_{\mathbf{g}}/\Gamma_{\mathbf{h}}$-orbit contained in $\psi^{-1}_{\mathbf{hg}}(z)$ that projects to that $\Gamma_{\mathbf{g}}$ orbit. So under the assumption that FIC1 fails, there is a $\mathbf{g}$, such that for every $\mathbf{h} \supseteq \mathbf{g}$ there are distinct $\Gamma_{\mathbf{g}}/\Gamma_{\mathbf{h}}$-orbits O_1, O_2 contained in $\psi^{-1}_{\mathbf{hg}}(z)$ that project to the same $\Gamma_{\mathbf{g}}$-orbit.

By Lemma 4.3.10, if two points are Galois equivalent they realize the same quantifier free τ-type; so O_1, O_2 realize distinct Galois orbits (and so any two orbits that project to them must realize distinct τ-types). But since $\overline{\Gamma}$ acts transitively on each $Z_{\mathbf{g}}$, there is a complete tree of splittings of $\mathrm{Aut}(\mathbb{C}/\mathrm{L})$ orbits that all project to z. This contradicts Keisler's theorem. So $\aleph_1$-categoricity of T^{∞}_{SF} implies FIC1.

Remark 4.4.16. [DH17, §5], using both Serre's open mapping theorem [Ser72, §6] for the finite index condition and work by [Rib75] on Shimura curves show FIC1 and FIC2 hold for all modular and Shimura curves. So our remaining sections concern higher dimensional varieties. FIC1 is known for some higher dimensional Shimura varieties and conjecturally for others, while FIC2 is true for all [Ete22].

[DH17] use both to prove categoricity. Since the Galois group is not accessible in our formal language, FIC1 cannot be directly expressed in the two-sorted theory. So the goal of a 'fully formal invariant' cannot be achieved unless explicit reliance on the finite index conditions as an hypothesis is avoided.

5 First order Excellence

Here is the opening paragraph of [BHP20].

> *Let $\mathbb{G} = \mathbb{G}^n$ be a complex algebraic torus, or let $\mathbb{G}$ be a complex abelian variety. Considering $\mathbb{G}(\mathbb{C})$ as a complex Lie group, with $\mathbf{L}\mathbb{G} = \mathbf{T_0}(\mathbb{G}(\mathbb{C}))$ its (abelian) Lie algebra, the exponential map provides a surjective analytic homomorphism*
>
> $$\exp : \mathbf{L}\mathbb{G} \twoheadrightarrow \mathbb{G}(\mathbb{C}).$$

In the spirit of Zilber, their paper aims at finding 'algebraic descriptions' of the cover $\exp$ which characterize the standard structure (at least up to categoricity in power). They solve a more general problem by providing a first order theory $\hat{T}$ for the situation and showing each model $\tilde{M}$ ($\hat{M}$ here) of $\hat{T}$ is determined by relations among two designated substructures and a certain transcendence degree. In this generality, the result is proved for any abelian group of finite Morley rank (henceforth fmr groups). Then, under slightly stronger hypotheses, the result becomes a true categoricity result for, in particular, an abelian variety defined over a number field.

We address in this section four new ingredients: formalized non-standard covers, 'first order excellence', Kummer theory, and a distinction between classification and categoricity. First order excellence appears to be both necessary and applicable for higher order Shimura varieties.

As noted in [BHP20], the quasiminimal approach studied earlier in this paper suffices to prove the $L_{\omega_1,\omega}$-categoricity in power for Abelian varieties. The goal of this section is to identify the distinctive elements of the [BHP20] proof that later reappear in [Ete22].

5.1 The two-sorted structure and fmr groups

A first order theory T is stable in κ if any $M \models T$, with $|M| = \kappa$, $|S(M)| = \kappa$. ($S(M)$ denotes the set of 1-types over M.) Morley showed that ω-stability (more properly, $\aleph_0$-stability) of a theory T is equivalent to stability in all powers (and also to the Morley rank having an ordinal value for each type). We need here a slightly weaker condition called *superstability*: T is stable in κ if $\kappa \geq 2^{\aleph_0}$.

The theory of $(\mathbb{Z}, +)$ is one of the prototypical strictly superstable theories[6] (that is, superstable, but not $\aleph_0$-stable). One can fix arbitrarily the congruence class of an element x for each n. This gives $2^{\aleph_0}$ distinct types realized by non-standard integers.

There is an extensive theory of fmr groups (see [BN94, ABC08]). We need here only the basics. In particular, Macintyre's result [Mac70] that an ω-stable group is divisible by finite. We now introduce the two-sorted theory; with that notation we are able at the end of this section to outline the main steps of the proof.

Unlike [DH17] where $\varprojlim Z_{\mathbf{g}}$ is in the background of the proof of (our) Theorem 4.4.12 but not the statement, [BHP20] build the structure of non-standard covers into the vocabulary of the two sorted structure by the ρ_n below.

[BHP20, §2.2] use the inverse limit of Definition 5.1.1 for *divisible* abelian groups; although it is not profinite, they refer to it as a profinite universal cover denoted $\hat{G}$ of G and G is renamed as M. Although the hat has only one meaning in [BHP20], it becomes overloaded here so we denote the inverse limit defined below as $\tilde{M}$. While in [BHP20] a typical 2-sorted (3-sorted in §8) structure $\hat{\tau}$ is represented as either $(\tilde{M}, M)$ or $\tilde{M}$, we write $\hat{M} = (\tilde{M}, M)$ and $\tilde{M}$ for the or (profinite cover) inverselimit from [BHP20, 1.2, 2.1] as that is the actual usage in most of the cited paper.

Definition 5.1.1 ($\tilde{M}$). *Given a commutative, divisible, abelian group $(M, +)$, consider the inverse limit $\tilde{M} = \varprojlim M_n$ of isomorphic copies M_m of M with the index set partially ordered by $m \leq n$ if and only $m|n$ and with maps η_{nm} (multiplication by*

[6] The other one is the theory of countably many equivalence relations E_n such that for each n, each E_n-class is split into infinitely many E_{n+1}-classes (and $E_{n+1} \subseteq E_n$).

$\frac{m}{n}$) taking $M_n \mapsto M_m$. Concretely, $(\tilde{M}, +)$ is the subgroup of the direct product of ω copies of M, containing those sequences ($\langle g_k : 1 \leq k < \omega \rangle$) such that if $k = nm$, $g_m = n \times g_k$ and $g_n = m \times g_k$.

Notation 5.1.2 (The vocabulary $\hat{\tau}$)**.** *Let $\mathbb{G}$ be the given abelian group and $T := \mathrm{Th}(\mathbb{G})$ in a large enough countable language that T has quantifier elimination. Further, let $\hat{T}$ be the theory of $(\hat{\mathbb{G}}, \mathbb{G})$ in the two-sorted language $\hat{\tau}$ consisting of the maps $\rho_n : \hat{\mathbb{G}} \to \mathbb{G}$ for each n, the theory T and, for each $\mathrm{acl}^{eq}(\emptyset)$-definable subgroup H of G, a predicate H for H and a predicate $\hat{H}$ for $\{x \in \hat{G} : \rho_n(x) \in H, n \in \mathbb{N}\}$.*

Although the kernel of $\rho = \rho_1$ is definable in the vocabulary given, a further predicate $\ker^0$ is included denoting the divisible part of the kernel (otherwise, it is only type-definable).

The axioms [BHP20, 2.5] of $\hat{T}$ are chosen so that

Theorem 5.1.3. *[BHP20, 2.7, 2.8, 2.21] For an fmr group $\mathbb{G}$, $(\underleftarrow{\mathbb{G}}, \mathbb{G}, \rho_0) \models \hat{T}$ and therefore $\hat{T}$ admits quantifier elimination and is superstable of finite U-rank.*

Although the T in Notation 5.1.2 is ω-stable, $\hat{T}$ is only superstable; also, many elements of $\ker(\rho)$ are not divisible in $\ker(\rho)$.

Remark 5.1.4 (Quasiminimality, unidimensionality, notop)**.** Abelian varieties as opposed to fmr groups, can be handled either by the quasiminimality methods of Section 4 or by the methods described in this section. A crucial distinction from Section 4 is that the former considered only the theory of unary functions from a group acting on the domain, while here we have the full group structure.

To explain the fmr proof we need some further model theoretic background. In general two types p, q over M are *orthogonal* when in different models N extending M the number of realizations of p and q can be varied arbitrarily. *Non-orthogonality* for strongly minimal sets has a particularly clear meaning. The strongly minimal sets D_1 and D_2 are non-orthogonal if there is a definable finite to finite binary relation on $D_1 \times D_2$. A theory is unidimensional if all types are non-orthogonal.

The three features that underlie the [BHP20] proof are.

1. A fmr abelian group has *finite width* [Bal88, XV.1] (aka almost $\aleph_1$-categorical [Las85]): Any model is the algebraic closure of the union of the bases of a collection of strongly minimal D_i for $i < n < \omega$. The D_i are defined over the prime model (the unique up to isomorphism model elementarily embedded in every model of the theory).

2. In models of $\hat{T}$ with M_0 the prime model of T and where $\mathbb{G}$ is defined over a number field k_0, Kummer theory allows the control of $\rho^{-1}(M_0)$ by the kernel $\rho^{-1}(0)$.

3. In studying Abelian varieties the n in 1) can be taken as 1 because the variety is interalgebraic with an algebraically closed field and so *almost strongly minimal* ($M = \mathrm{acl}(D)$ for strongly minimal D).

Since Kummer theory doesn't apply to arbitrary Shimura varieties, both 2) and 3) fail for more general higher dimensional Shimura varieties (see Section 6).

5.2 First order Excellence and fmr groups

Shelah's main gap program defines a sequence of properties X of countable first order theories forming a sequence of dichotomies [Bal18, §5.5] such that: if T satisfies X, T has the maximal number of models in every uncountable cardinal. If T fails X, the models of T satisfy conditions useful for classification. (e.g. stability implies the existence of the 'non-forking' independence relation). The positive side of the final dichotomy in the sequence is superstable *without the omitting types order property* (denoted **notop**). Under this hypothesis, Shelah ([She90] and earlier papers) showed that an appropriate class of models of T had a notion of independence among structures with n-amalgamation for all n that yields the classification of models. Hart [Har87] reduced the amalgamation requirement to 2-amalgamation and this reduction was extended to the quasiminimal excellent case in [BHH$^+$14]. In Section 6, we note this 'notop' approach is used to study higher dimensional Shimura varieties.

In Section 3 of [BHP20] the techniques of [Har87] are adapted to the specific framework here to establish a decomposition of models of $\hat{T}$ analogous to that in Remark 5.1.4 for models of T. This yields

Theorem 5.2.1. *[BHP20, Theorem 3.31] Each model $\hat{\mathbb{M}}$ of $\hat{T}$ is determined up to isomorphism by the transcendence degree of the algebraically closed field K such that $M \cong \mathbb{G}(K)$, the isomorphism type of the inverse image, $\hat{M}_0$, of the prime model M_0 of T, and the isomorphism type of M over M_0.*

5.3 Abelian Varieties

From the model theoretic standpoint, an *Abelian variety* is a complete algebraic variety whose points form a group such that the group operations are definable in the ambient field. For Abelian varieties, Kummer theory eliminates (as in [Gav08, BGH14]) the reliance in Theorem 5.2.1 on knowing the isomorphism type of $\check{M}_0$ over the kernel. The situation described in the opening paragraph of § 5 is a special case. Namely, let $\mathbb{G}$ be (the formula defining) an abelian variety $\mathbb{G}(K)$ over a field K as in the introduction to Section 5. Assume $\mathbb{G}(\mathbb{C})$ and its ring of endomorphisms are definable over a number field k_0. With this notation:

Theorem 5.3.1. *[BHP20, Theorem 4.6] a model $\hat{M} = \langle \tilde{M}, M, q \rangle$ of $\hat{T}$ is determined up to isomorphism by the transcendence degree of the algebraically closed field K such that $M \cong \mathbb{G}(K)$, and the $\hat{\tau}$ isomorphism type of* $\ker \rho$.

Remark 5.3.2 (Complete formal invariant)**.** Theorem 5.3.1 gives categoricity in all uncountable cardinalities by adding the $L_{\omega_1,\omega}$ sentence characterizing the standard kernel. But Theorem 5.3.1 is more general than categoricity; it shows that models with non-standard (possibly uncountable) kernel are characterized by the $\hat{\tau}$-diagram of the kernel. Of course, this statement cannot be formalized in languages with bounded length of conjunctions since the kernels can be arbitrarily large. But Zilber's goal (just after Notation 1.0.1) only aimed at complete formal characterization for prototypical mathematical structures.

6 Higher Dimensional Shimura Varieties

A Shimura variety is a higher-dimensional generalization of a modular curve that arises as a quotient variety of a Hermitian symmetric space X^+ by a congruence subgroup of a reductive algebraic group defined over $\mathbb{Q}$. We consider Shimura varieties that are moduli spaces for generalized algebraic varieties. Rather than discussing further technical details on the definition of a Shimura datum (G, X), we survey the differences that arise in generalizing the results in Remark 4.4.16 about Shimura curves to higher dimensional Shimura varieties: $S(\mathbb{C}) = \Gamma \setminus X^+$.

Central difficulties arise directly from the higher dimension in two ways. First, in the curve case the 2-sorted structure is (almost)-quasiminimal because the variety in field sort is a curve and so strongly minimal and the geometric closure on the cover sort is given by $a \in \mathrm{cl}(X)$ if $a \in q^{-1}(\mathrm{acl}((q(X)))$. Quasiminimality can fail in the higher dimensions. Second, rather than special points which are fixed points of some g, one must treat *special subvarieties* [Ete22, §3.4] and finite unions thereof, *special domains*. The fact that these are not merely points leads to several difficulties.

1. The structure of the covering sort is no longer strongly minimal. Even after naming the elements of the group the special subvarieties give a complicated structure on the covering sort.
2. In the curve case the intersection of special domains was a point; that may fail in higher dimensions.
3. The theories of two inverse limit structures $\hat{\mathbf{p}}$ and $\tilde{\mathbf{p}}$ are considered as the covering space. The first structure is the analog of $\varprojlim Z_{\mathbf{g}}$ (Definition 4.2.11). The second consists only of the standard points of this limit. The canonical universal cover $\mathbf{p}$ satisfies the first order $\mathrm{Th}(\tilde{\mathbf{p}})$ but not in general $\mathrm{Th}(\hat{\mathbf{p}})$ [Ete22, Example 5.7, Corollary 5.14].

4. An $L_{\omega_1,\omega}$ categorical axiomatization is not claimed. Each model can be precisely characterized but the characterization is not in $L_{\omega_1,\omega}$. See Remark 5.3.2.
5. Finally, even this characterization depends on whether the variety under consideration satisfies *finite index conditions* as in the modular case. Although FIC1 and FIC2 are true in the modular curve case, here the truth of $FIC1$ for $\mathbf{p}$ is actually equivalent to the characterizability of models of $T^{\inf}_{SF}(\mathbf{p})$ since [Ete22] shows FIC2 is true.

7 Model Theory and Analysis

One can signal three different model theoretic approaches to analysis:

1. **Axiomatic analysis** studies behavior of fields of functions with operators but *without* explicit attention in the formalism of continuity but rather to the algebraic properties of the functions. The function symbols of the vocabulary act on the functions being studied; the functions are elements of the domain of the model.

   Example: DCF_0 as discussed below.
2. **Definable analysis** has a lower level of abstraction; the domain of the functions remains the universe of the model. The functions being studied are the compositions of the functions named in the vocabulary; one cannot quantify over them.

   Example: o-minimality.
3. **Implicit analysis** Attempts to provide 'algebraic characterizations of important mathematical structure by axiomatizations in infinitary logic that are categorical in power. Example: the material in this paper.

The first two are discussed in [Bal18, §6.3]. The work expounded in this paper has many commonalities with a prime example of axiomatic analysis: the study of transcendence results for solutions of differential equations by the study of the ω-stable theory DCF_0 of differentially closed fields of characteristic zero. The notion of 'not integrable by elementary functions (Painlevé said 'irreducible') is formalized by 'the solution set is strongly minimal' [Nag14]. The study of Schwartzian equations provides a general framework in which the j-function and modular curves are explored. The work includes, variations on the Ax-Lindemann-Weierstrass theorem, proofs that Generic differential equations are strongly minimal [DF23] and Differential Chow Varieties are Kolchin-constructible [FLS17], and analysis of strongly minimal solution sets defined by differential equations in terms of the Zilber trichotomy and $\aleph_0$-categoricity.

But while the mathematical topics are the same, the aims are different: The covers project tries to assign a categorical description of each cover. The DCF_0 approach tries to understand transcendence results for solutions of the differential equations.

The crucial methodological difference is the two-sorted nature of the cover program. The axiomatic analysis framework is preserved in that there is no explicit treatment of convergence or continuity. But connecting the domain and target by quotients under an explicit group action as well as the use of infinitary logic provides tools not available in the earlier examples of axiomatic analysis.

8 Families of covers of algebraic curves

In recent work Zilber and Daw [DZ22b, DZ22a] deal with *families* of covers of curves. They build on earlier constructions we have discussed in this paper. Rather than a cover of a *single* variety, albeit one that parameterized a family of varieties, an entire family of such covers is studied and the covering space becomes an *analytic Zariski structure* [Zil10]. In [Zil22] the analysis of families is generalized by being placed in a geometric algebraic setting.

The most salient difference between these works and those discussed earlier in this paper is that, rather than a cover of a *single* variety, an *entire family of covers* is now the main subject. Our earlier Definition 4.2.9 is now replaced by a basic vocabulary consisting of *three* sorts, together with maps $\Gamma_N \setminus \mathbb{H} \mapsto \mathbb{C}$ covering a family of curves $S_N(\mathbb{C})$.

8.1 Pseudo-analytic covers of modular curves

Major differences of paper [DZ22b] from the earlier discussion of modular curves include:

1. The basic vocabulary is now 3-sorted. More specifically, [DZ22b] considers structures $(D, G, j_N, \mathbb{C})$ where the $j_N : \mathbb{H} \twoheadrightarrow S_N(\mathbb{C})$. The discrete group is now given as a third sort incorporating *a group operation* (so its pregeometry is locally modular, rather than trivial). This sort contains group with distinguished subsets[7] $(\mathrm{GL}_2^+(\mathbb{Q}), \times, \mathrm{SL}_2(\mathbb{Z}), E(\mathbb{Q}), \{\mathbf{d}_q, \mathbf{d}'_q : q \in \mathbb{Q})\}$, where E is the collection of elliptic elements of the group; those that have unique fixed points. This structure is specified up to isomorphism by a sentence of $L_{\omega_1,\omega}$. But not all group elements are still named in the formal language.
2. The uniformizing functions j_N each map into $\mathbf{P}^3(\mathbb{C})$ rather than into the arbitrarily high dimensional spaces of the maps $[\phi_{\mathbf{g}}]$ in [DH17, Ete22]. Furthermore, these are now defined over $\mathbb{Q}$ rather than over $E^{ab}(\Sigma)$.

[7] E is the elliptic Möbius transformations and the $\mathbf{d_q}, \mathbf{d_q}$ are specific *diagonal* matrices.

3. As well as an almost quasiminimal axiomatization of the 3-sorted structure, the domain is considered as a Zariski Analytic set with a quasiminimal geometry. Both of these structures are shown to be uncountably categorical.
4. The special points *are not named*. However as in Definition 4.3.5 they are uniquely associated with elliptic elements of the group.

In many ways, this last distinction is the most important for the general program, as naming of the special points trivializes some of the arithmetic. In [DZ22b], the structure of the *family* is proved to be categorical in all uncountable cardinalities.

8.2 Locally o-minimal covers of algebraic varieties

The paper [Zil22] takes a *more general* approach. It abstracts away from naming all elements of the discrete groups as earlier in this paper. The relations among the universal and finite covers are given more abstractly as properties of maps from a domain (whose smoothness is defined topologically and geometrically but not algebraically) onto families of algebraic varieties. This smoothness as well as the eventual quasiminimality for curves[8] is controlled by *external* o-minimal structures.

Remark 8.2.1. 1. The formalization is new. For a fixed model R of the theory T of a fixed o-minimal expansion of the reals (e.g the restricted analytic functions) a structure $\mathbb{U}(\mathsf{R})$ is defined. The resulting structure $\mathbb{U}(\mathsf{R})$ is an abstract Zariski structure[9].

2. Generalizing the last paragraph of Section 8.1, in the standard model the domain is a complex manifold $\mathbb{U}(\mathbb{C})$ with holomorphic maps f_i onto algebraic varieties $X_i(\mathbb{C})$ with natural projections $\mathrm{pr}_{i,j}$ among the X_i. These analytic properties are definable using theory of K-analytic sets in o-minimal expansions of the reals developed in [PS08, PS10]. We fix $k \subseteq \mathbb{C}$ a subfield over which the varieties $\mathbb{X}_i$ are all defined.
3. The ostensibly two-sorted structure of 1) becomes one-sorted because the field can be interpreted in the abstract Zariski structure. And the third sort of Section 8.1 has disappeared because the group is no longer referenced directly.
4. The o-minimal geometry of algebraic closure in $\mathbb{U}(\mathsf{R})$ imposes the desired quasiminimal geometry on $\mathbb{U}(\mathsf{R})$. The dimension function is denoted cdim for 'combinatorial dimension'. Note that the ordering is not externally imposed on $\mathbb{U}$: rather, it is implied by the predicates described in (1) above and the dimension just mentioned.

[8] The set-up is for arbitrary algebraic varieties, but the categoricity result is only for curves and we restrict to that case.

[9] Actually, $\mathbb{U}(\mathsf{R}) = U(K)$ where K is taken as an algebraically closed field $\mathsf{R} + i\mathsf{R}$ and $U(R)$ is constructed analogously to $U(\mathbb{C})$.

5. As before, there is an $L_{\omega_1,\omega}$ sentence that axiomatizes the quasiminimal (excellent) geometry and whose models form an AEC that is categorical in all cardinalities.

Zilber provides a proof of the following theorem [Zil22]:

Theorem 8.2.2 (Categoricity of families of smooth complex algebraic varieties [Zil22])**.** *Let* $\mathbb{U}$ *be a cover of a family of smooth complex algebraic variety, formalized as in Remark 8.2.1, and let* $\mathfrak{U}(\mathsf{R})$ *be its associated* $L_{\omega_1\omega}$*-definable class. If* $\dim_{\mathbb{C}}(\mathbb{U}) = 1$*, (i.e. if the varieties are curves) and* $\mathrm{cdim}(\mathsf{R}/k)$ *is infinite, then* $\mathfrak{U}(\mathsf{R})$ *is categorical in all uncountable cardinals.*

Zilber remarks that in the case of higher dimensional varieties, categoricity in $\aleph_1$ can still be proved.

Example 8.2.3. Here are some examples from [Zil22]. Fix the o-minimal expansion $\mathbb{R}_{\mathrm{An}} = \mathbb{R}_{\exp,an}$ of the reals with the exponential function and the restricted (to bounded intervals) analytic functions.

- Let $I = \mathbb{N}$, $\mathbb{U} = \mathbb{C}$, $f_k(z) = \exp(\frac{z}{k})$, $D_n = \{z \in \mathbb{C} : -2\pi n < \mathrm{Im}(z) < 2\pi n\}$. These are easily seen to provide a cover system.
- The j-function with variants j_N as uniformizers for the modular curves $\Gamma_N \setminus \mathbb{H}$ are examples; this study allows one to formalize their analytic properties in terms of o-minimality. Finally, other examples include the Siegel half-space and polarized algebraic varieties (these last examples are claimed but not developed by Zilber).

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