LOGIC ACROSS THE HIGH SCHOOL CURRICULUM

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Here are some examples of fundamental issues that arise in high school mathematics. Students are puzzled by the assertion $.\overline{9} = 1$. After giving an excellent presentation[1] concerning her extensions to the logic unit in a geometry course, Teacher A asked, 'How do I respond when a student asks if it is true that the sum of the angles of a triangle is $180°$'? A freshman complains, I learned in primary school that you couldn't take away a bigger number from a smaller one, what are you doing? A junior objects that every step in his solution of

$$2 + \frac{1}{x-2} = \frac{x-1}{x-2}$$

is correct. Why is his answer of 2 wrong? A trigonometry text book poses the question, 'Show $\sin A = \sin B$ if and only if $A = B + 360K$ or $A + B = 180 + 360K$.' What does this mean?

Full answers to these questions depend on the logical analysis developed during the twentieth century. The central activity of the Chicago Teacher Transformation Institutes is the design and delivery of a sequence of courses for secondary teachers. These are to be graduate courses in mathematics acceptable for degree credit tailored for secondary teachers. 'Logic across the high school curriculum' was conceived as a version of UIC's Math 430 adapted for secondary teachers. In the remainder of this note, I develop in more detail the rationale for such a course and some details of 'tailoring'. Logic is a specific strategy for metacognition, 'the ability to monitor ones current level of understanding and decide when it is not adequate' (page 47 of [4]) and metacognition is a key element in how people learn.

One goal of the CTTI courses is to acquaint teachers with recent work. It is very difficult to discuss current technical developments in mathematics. However, three of the Time magazine list of the 20 most important scientists and thinkers of the twentieth century were logicians and the work of two of them (Gödel and Turing) deeply impacts our understanding of high school mathematics in a way accessible to teachers. Mathematicians have generally internalized these understandings of mathematics, but, perhaps without exploring the rationale. The aim of this course is to short circuit the generalization to mathematical maturity from mastering technical material in a number of areas in mathematics by focusing on the way mathematics is organized and justified, that is on logic.

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[1] I am recalling the gist of a conversation after a contributed talk several years ago at the Chicago Symposium in Teaching and Learning.

This note outlines the themes of a course meeting two goals for a CTTI course: connecting directly to high school mathematics and exposing teachers to recent highly significant mathematics.

1. A FUNDAMENTAL DISTINCTION

The first step to answering Teacher A's question is a following fundamental distinction made by Aristotle.[2]

Deduction is the process of extracting information implied by given premises regardless of whether those premises are known to be true or even whether they are true. A deduction from premises whose truth value is not known produces knowledge - of the fact that its conclusion is a consequence of its premises – not just from knowledge of the truth of its conclusion.

Demonstrative logic is the study of demonstration (conclusive proof) as opposed to persuasion or even probable proof. Demonstration produces knowledge. According to Aristotle, a demonstration is an extended argumentation that begins with premises known to be truths and that involves a chain of reasoning showing by deductively evident steps that its conclusion is a consequence of its premises. In short, a demonstration is a deduction whose premises are known to be true. Mathematics is concerned with deduction and not with demonstration. So the short answer to whether the sum of the angles of triangle is $180°$ is true, is 'That is not a mathematical question; it asks for a demonstration, whether a statement is true, not a deduction, whether it follows from hypotheses.'. But a deeper answer is that we must be more clear about what is meant by 'true'. And that answer will be developed through this essay.

The distinction between demonstration and deduction is particularly important for geometry. Until the 19th century it was thought that geometry was the deduction of truths from *unassailable* premises that described the physical world. These premises were Euclid's Axioms (common notions) and Postulates (geometric assumptions). We now take geometry as the deduction of conclusions from a certain set of geometric hypotheses. These hypotheses might allow exactly one, many or no parallel lines. Whether these geometrical hypotheses are "true" is *not* a mathematical question. But whether specific geometric statements follow from particular sets of hypotheses is the essence of mathematics. So is formulating a notion of truth so that such a verification justifies saying the proposition is true (somewhere). We need further concepts.

It may be particularly relevant, given the science/math focus of the CTTI, to discuss the distinction between justification in mathematics and science.

2. FORMAL LANGUAGE

Teacher A's simplest correct and informative reply to the question about the sum of the angles of a triangle is, 'It is true in the systems we are studying'. We now explore at some length what that means. In the course of that exploration we discuss many other basic notions of high school mathematics that are clarified by

[2] These comments are adapted from unpublished notes of John Corcoran.

this approach. We have to consider what a mathematical statement is; we first focus on algebra.

The key idea of mathematical logic is the distinction between syntax, a formal language for expressing mathematics, and semantics, mathematical structures where sentences of this language are true or false. A second key component of syntax is the notion of proof in a formal system. The *formal* proof notion is far more pertinent to high school mathematics than to college or certainly to most mathematical research. Mathematical research is about the relations among mathematical structures; high school algebra is about formal derivations in the language of ring theory.

The formal language makes it easy to distinguish two fundamentally different mathematical usages: terms that denote numbers (usually called expressions in high school algebra) and relations between numbers (statements that are true or false). We now discuss several other notions, fundamental to high school math, that are clarified by thinking of formal languages.

Equality and Equivalence relations. In a course for teachers, equality deserves special treatment for several reasons. First is to show that equality plays a common role in the various algebraic axioms systems and geometry rather than (as the language in American algebra texts suggests) new properties of equality are invented for each operation and each subject. (Texts refer to the addition property of equality and the multiplicative property of equality rather than just noting that addition and multiplication are functions.)

Second, while the distinctions between equality and equivalence are properly glossed over for K-12 students, secondary teachers need to understand the full story. Expressions in a formal language such as $\frac{3}{4}, \frac{6}{8}, .75, .74\overline{9}$ are equivalent representations of a single rational number. And, *pace* Frank Morgan, decimal representations are expressions in a formal language, they are not numbers[3].

In algebra, we prove $x^2 - 4 = (x + 2)(x - 2)$ to show each of the expressions determines the same function from $\Re$ to $\Re$. The notion of equivalence relation is central both to the idea of normal form, which permeates high school mathematics without explicit mention and to the notion of quotient structures that is not formally studied in high school but is important for a full understanding of the material.

Parsing. One of the most unpleasant parts of many logic books is the totally unmotivated proof on page 2 that the expressions in the formal language satisfy the 'unique readability' property. This is often accomplished by a sophisticated induction on the number of parentheses, or by studying Polish or inverse Polish notation. Who cares? Any teacher who mistakenly drills students on PEMDAS cares.[4] But Pemdas is flawed because it tries to provide a simple rule where no simple rule is possible. The goal for high school teaching should be to understand the role of parentheses and the role of the associativity and commutativity in sometimes omitting them. Clear statements of 'simplification' requires both the notions

[3]In [7], the construction of the real numbers is avoided by a one sentence reference to infinite decimals.

[4]'Please Excuse My Dear Aunt Sally' is a flawed mnemonic for teaching order of operations.

of derivation and equivalence relations. Study of this concept focuses attention on the basic pedagogical issue: 'simplification for what'.

The definition of a polynomial is the simplest example of the inductive definition of a term (expression) in a formal language.

Definitions. Students need to understand that a mathematical definition is just an abbreviation for certain relations among concepts that are currently being considered. That is, we consider explicit definition: 'a is prime' if and only if 'its only divisors are itself and 1. More formally we can introduce a new predicate P for prime numbers and write $\mathrm{P}(x) \leftrightarrow (y|x \leftarrow y = 1 \vee y = x)$. It makes no sense to ask if a definition is true, but it is often important to ask whether two definitions are equivalent.

Variables. As we discuss in Section 4, most calculations in high school algebra use quantifier free formulas. But already in problems like [5] 'Show $\sin A = \sin B$ if and only if $A = B + 360K$ or $A + B = 180 + 360K$,' there are treacherously suppressed uses of the existential quantifier. Not to mention that the statement fails a basic logical test of meaningfulness: equivalent statements have the same free variables.

The only coherent explanation of the notion of variable involves the interpretation of formal languages in structures. In the next section, we discuss this notion, formalized by Tarski in 1933 but in common use by mathematicians long before. This substitutional account of the meaning of variable is pervasive in high school mathematics. This approach to the notion of variable occurs in all logic texts; it is laid out in the context of high school algebra in [1].

This approach to variables allows a uniform understanding of solution of equations, variables as arguments for functions, analytic geometry, the role of parameters and families of functions; these are all crucial topics of high school mathematics. Teachers (and many texts) complicate the curriculum because they don't understand the essential unity.

This framework enables the discussion of the difference between intensional (function as rule) and extensional (function as set of order pairs) definition of function[6].

The study of calculus and limits require quantification. This is precisely what led Newton down the false trail of 'variable quantities'. And by studying quantification we are able to give a clearer and more full explanation of limits and the foundations of calculus.

3. NUMBER SYSTEMS, TRUTH, AND VALIDITY

A student learns in primary school that he can't take away a bigger number from a smaller one. But in the upper grades, he can. The student is now working in a different number system. But has that distinction been made strongly enough? It does not make sense to speak of the truth of a statement without specifying

[5] This is a direct quote from a text whose name we suppress.

[6] U.S. teachers who went to high school in the late 20th century were taught only the extensional definition; now they are expected to teach the intensional version.

the system about which it is asked. Thus, the primary school understanding of subtraction is correct for the natural numbers but not for the integers.

Number Systems. A structure or model for a formal language is a set with a prescribed group of operations corresponding to the relation and function symbols in the language. This is the general term exemplified by the concept of a number system.

Note that the formal language of ring theory (say two binary operations of addition and multiplication, a constant 0, and a unary $-$) is applicable to a wide variety of structures: Boolean algebras, the integers, the rational numbers, and the real numbers. By changing the formal language similar ideas apply to geometry. A geometry is a system of points, line, planes and and relations of incidence and congruence. In the Birkhoff-Moise formulation it also includes the real numbers. The exact formulation is irrelevant to the insight that both number systems and geometries are models for statements in a formal language. And the question 'in which geometries is the sum of the angles equal to 180^o?' is of the same type as 'in which fields is there a square root of two?'.

This viewpoint provides a connection between the subjects of high school mathematics from basic algebra through geometry to trigonometry and the beginnings of calculus.

Number systems can be approached both intuitively – everyone has some understanding of the natural number sequence – and formally. Formally can be understood in two ways. Specifying an axiom system (which specifies the structure uniquely?) or constructing a system from more primitive systems. Thus we can define the reals as the unique complete ordered field. Or the reals can be constructed as completions (by adding cuts or summing series) of the rational field. Under either definition, the theory of limits can be developed to justify the assertion that the equation $.\overline{9} = 1$ holds in the real numbers.

Truth. There is a systematic way to define the concept that a sentence ϕ is true in a system A for the language of ϕ. We write $A \models \phi$. Note that *truth* is defined only in particular systems. A sentence that is true in all systems for a given vocabulary is *valid*. We move here from number systems to more general systems such as linear orders, geometries, or most concretely, Tarski's world. Tarski's world broadens the fairly intuitive idea of what it means for an equation to be true in the real numbers and enables a concrete understanding of quantification. This provides a framework for understanding truth of geometric statements in various models of geometry. Thinking of plane geometry as a system of point and lines with an incidence relation formalizes Hilbert's famous quip about chairs, tables, and beer mugs.

4. Proof

Why can a student do each step correctly in 'solving'

$$2 + \frac{1}{x-2} = \frac{x-1}{x-2}$$

and obtain an incorrect answer? He has not understood that the solution procedure is a proof that identifies the only possible solutions, but that it remains to be determined if the identified numbers are solutions – that is do they satisfy the equation? This is a complex situation. The isolation of 2 as the only possible root is a derivation in the formal system of ring theory, which provides solutions in the case of polynomial equations. But the student has overextended the method. For polynomials, the isolation of possible roots is literally a formal proof in equational logic as discussed in undergrad texts by Burris or Barwise-Etchemendy. But solving the equation is rather more complicated: it requires knowing the definition of 'solution'. The 'check' is the essential part of the solution.

Logical Consequence. A proposition ϕ is a *logical consequence* of a proposition ψ if whenever ψ is true, ϕ is true. This fundamental notion is often as expressed as saying the inference from ψ to ϕ is valid. The difficulty with this notion is that, at least *a priori*, it is uncheckable. How do we analyze the 'whenever'? We have spoken of language and models but not of proof; proof is a fundamental tool to relate statements and their truth.

Durand [3] analyzes several notions of logical implication and notes that student confusion between the truth conditions for $(\forall x)[P(x) \to Q(x)]$ and $P(a) \to Q(a)$ may underlie what appear to confusions between implication and equivalence. She notes difficulties caused by the imprecise use of quantifiers in high school. These studies support the careful study of the logic of atomic sentences as a prelude for first order logic.

Formal Proof. Almost all logic books introduce some formal proof system, with arcane, often unmotivated, rules, and frequently little connection to proofs as mathematicians do them. Books on mathematical reasoning often introduce various rules of inference and explain their connections to how mathematicians actually prove theorems. Both approaches lose a big idea: There is a small list of reasoning principles, which can be exhibited in the course, that account for *all* correct (valid) inferences. The set of validities is unique; the choice of reasoning principles is not. Each text book provides a *different but equivalent* set of reasoning principles.

Theorem 4.1 (Godel). *There is a set of axioms and rules of inference for first order logic such that a sentence is true in every structure if and only if it can be proved in this system.*

Or in the language of the last subsection, a sentence is valid if and only if it provable.

Let us distinguish between proof, which provides both explanation and justification, and derivation, which provides only justification. No one except computers and beginning students derives, that is, does proofs in a formal system. Why is this relevant for teachers? Confusion between these notions lead to an unhealthy focus on extremely formal arguments in algebra/geometry.[7] The goal of this course

[7]The first 'proof' for many American high school students is a six step excursion passing through axioms of the real numbers to show that if equal segments are taken away from equal segments the remaining segments are equal.

is for teachers to prove not derive. But more important, to clarify the difference. The reason these notions are often confused is because, as noted at the beginning of this section, the 'proofs' that are most common in high school algebra are often misconstrued as simply derivations.

5. THEORIES

We return to Teacher A's reply to the question about the sum of the angles of a triangle, 'it is true in the systems we are studying'. In the Section 3, we elaborated on the notion of system and in particular systems for geometry. But how does one specify the ones we are studying? Fix a set of postulates. The statement is true in any geometric system satisfying appropriate axioms. But it is not valid. It is a theorem of the theory given by those axioms. By studying theories, we can provide both a unifying theme for high school mathematics and introduce the teachers to the inspiring results of 20th century logic.

Axiomatics. The powerful ideas that ought to unify high school mathematics, distributivity and commutativity, the existence of inverses, etc. are instead presented as new ideas in each separate field leaving the student with hundreds of unrelated facts to connect. By exhibiting these as general laws which apply to integers, rationals and reals (for appropriate operations) the essential unity can be demonstrated. Understanding the connections among the first order theories of integral domains, fields, the real field, and the complex field provides a coherence to the high school curriculum. In particular, the notion of algebra as generalized arithmetic is justified as a key tool for the teaching of introductory algebra.

In geometry, the distinction should be made between the local proof in some curricula (CME, IMP) and the global axioms of Moise-Birkhoff which dominate American high school geometry. Background in formal language enables the role of non-standard models and different axioms systems for geometry to be understood.

Completeness in high school algebra A simpler form of Gödel's completeness theorem is endemic in high school algebra. Any equation between polynomials that is true in the real numbers is provable from the equational axioms for a commutative ring with multiplicative inverse. Tarski's high school algebra problem asked whether the same result held for exponential polynomials (as studied in the second algebra course). Surprisingly, the answer is no; the counterexample is displayed in [8]. A positive result is more germane to high school. Students are asked to solve over the reals a polynomial inequality in one variable. What does that mean? The set of real numbers that satisfy the inequality can be represented as a Boolean combination of intervals. There is a hidden theorem that every system has a solution of this form. There are important generalizations of this idea. Any ordered structure where any first order definable set (in particular an inequality) defines a Boolean combination of intervals is called o-minimal. Tarski showed the real field is o-minimal in the 1930's; Wilkie showed the real exponential field is o-minimal in 1996 [10]. The study of o-minimal theories resulted in a research community of real algebraic geometers and logicians with important applications in analysis.

Algorithms. At the elementary level, the distinction between algorithms for arithmetic operations (e.g. multiplication) and the basic function they compute must be made clear. But in the computer age the notion of algorithm takes on a much more central role. The very idea of a stored program computer arose with Turing's work in formalizing the abstract idea of computation. The distinction between definition by recursion and proof by induction often arises as teachers address the idea of modeling physical and geometric sequences by algebraic formulas. Yet such concepts as the distinction between open and closed forms for functions are often unknown to them. The distinction between unsolvability and infeasibility appears[8] in some high school curricula.

Incompleteness and Undecidability. Hilbert's first problem asked for a solution to the continuum problem; the second asked to prove the consistency of arithmetic; the tenth asked for a method to determine if an arbitrary family of Diophantine equations is solvable. The crucial insight of twentieth century logic is not just the solution of these problems but refining the understanding of what solution means, by providing a precise meaning for when a problem is not solvable. This led to the development of computability theory. The significance of this subject for society is indicated by the article 'An Explanation of Computation Theory for Lawyers' [9]. The mathematical part of this article is appropriate for this course.

The work of Gödel and Cohen shows that analogously to parallel postulate in geometry, the continuum hypothesis can be neither proved nor disproved from the accepted axioms for set theory (Zermelo-Fraenkel plus the axiom of choice). Gödel also showed (given Turing's analysis of computability) that there is no computable algorithm to determine whether a sentence of arithmetic is true in the natural numbers. Nor can the consistency of arithmetic be proved within the system, which is what Hilbert hoped.

We can return to the kind of questions with which we began. Euclidean geometry is incomplete in the sense that certain propositions are neither consequences nor refuted by the basic axioms. We can find a computable set of axioms for geometry such that their consequences are a complete theory (every sentence or its negation is provable). There is an algorithm that decides whether any given sentence is true in the standard model of geometry; there is no such algorithm for arithmetic.

To make the distinction concrete, let $p_1(x_1, \ldots x_n) = 0, \ldots p_k(x_1, \ldots x_n) = 0$ be a finite system of equations in n variables. There is a computable function that tells for each such system whether or not it has a solution in the real numbers. There is not (not just undiscovered, there is not) such an algorithm if the question is solution in the integers. Whether there is a such an algorithm for the rational field is a question investigated by logicians and number theorists of the highest level.

6. Conclusion

The course in logic across the high school curriculum provides a splendid opportunity to clarify teachers ideas about notions that are fundamental in their daily teaching while connecting these ideas to some of the most important results of

[8] or is confused

20th century mathematics. The examples cited above demonstrate that basic logical issues are particularly important for high school teaching as opposed to college teaching or for mathematical research. The actual connection of logic with research in core mathematics can be seen in such articles as ([2, 6, 5]); those methods are much more advanced than I discuss here.

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