Model Theory and the Philosophy of Mathematical Practice[1]
Formalization without Foundationalism

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Introduction

{introza}

The announcement[1] for a conference on Philosophy and Model Theory in 2010 began:

> Model theory seems to have reached its zenith in the sixties and the seventies, when it was seen by many as virtually identical to mathematical logic. The works of Gödel and Cohen on the continuum hypothesis, though falling only indirectly within the domain of model theory, did bring to it some reflected glory. The works of Montague or Putnam bear witness to the profound impact of model theory, both on analytical philosophy and on the foundations of scientific linguistics.

My astonished reply to the organizers[2] began:

> It seems that I have a very different notion of the history of model theory. As the paper at [**Bal10**] points out, I would say that modern model theory begins around 1970 and the most profound mathematical results including applications in many other areas of mathematics have occurred since then, using various aspects of Shelah's paradigm shift. I must agree that, while in my view, there are significant philosophical implications of the new paradigm, they have not been conveyed to philosophers.

This book is an extended version of that reply to what we will call *the provocation*[3]. I hope to convince the reader that the more technically sophisticated model theory of the last half century introduces new philosophical insights about mathematical practice[4] that reveal how this recent model theory resonates philosophically, impacting in particular such basic notions as syntax and semantics[5], structure, completeness, categoricity, and axiomatization. Thus, large sections of the book are devoted to introducing and describing for those not familiar with model theory such topics as the stability theoretic classification of first order theories (a generalization of vector space independence that gives a new notion of invariants of structures,

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[1]This delightful Paris conference was organized by Brice Halimi, Jean-Michel Salanskis, and Denis Bonnay. The full announcement is here. `http://dep-philo.u-paris10.fr/dpt-ufr-phillia-philosophie/la-recherche/les-colloques/philosophy-and-model-theory-314434.kjsp`

[2]Letter to Halimi, September 20, 2009.

[3]The announcement contains a number of astute observations that we will comment on in due course.

[4]See page xii.

[5]We use semantics for 'semantics of formal language' (page 209). Among the accomplishments of model theory alluded to in the provocation are contributions to the philosophy of language as in [**BW16**]; our topic here is the philosophy of mathematics.

which applies to various mathematical topics) and the discovery that these abstract properties imply the interpretability of classical groups (page 15) into apparently unrelated contexts. Much of this exposition will be in the context of discussing the paradigm shift[6]. In short, the paradigm around 1950 concerned the study of *logics*; the principal results were completeness, compactness, interpolation and joint consistency theorems. Various semantic properties of theories were given syntactic characterizations but there was no notion of partitioning all theories by a family of properties. After the paradigm shift there is a systematic search for a finite set of syntactic conditions which divide first order theories into disjoint classes such that models of different theories in the same class have similar mathematical properties. After the shift one can compare different areas of mathematics by checking where theories formalizing them lie in the classification. {paradigm shift}

Framework for formalization. We *always* speak of formalizing a *particular* mathematical topic. A *formalization* of a mathematical area specifies a vocabulary (representing the primitive notions of the area), a logic, and the axioms (postulates in the technical sense of Euclid) for the particular topic.

By a topic we might mean group theory or algebraic geometry, or perhaps set theory. We argue that comparing (usually first order) formalizations of different mathematical topics is a better tool for investigating the connections between their methods and results than a common coding of them into set theory. Avoiding a global foundation allows us to evade the Gödel phenomena and study instead {tame} different 'tame[7]' areas of mathematics: e.g. any stable or o-minimal theory, real algebraic geometry, differentially closed fields etc. In Chapter 1 we elaborate on this notion of formalization and explain the importance of each of the choices: vocabulary, logic, and axioms.

We develop some uses of first order formalization in studying the organization of mathematical practice, some consequences of this new organization for traditional mathematics, and explore how the analysis of formalization here affects some standard topics in the philosophy of mathematics. This book supports four main theses.

Theses

(1) Contemporary model theory makes formalization of *specific mathematical areas* a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).

[6]See further detail on page xiv. While I invoke Kuhn's term, I don't want to take on all of its connotations. My meaning is attuned with Harris (page 23 of [**Har15**]) "I soon found myself caught up in the thrill of the first encounter between two research programs, each of a scope and precision that would have been inconceivable to previous generations, each based on radically new heuristics, each experienced by my teachers' generation as a paradigm shift.'. He later gives Weil, Grothendieck, and Langlands as examples of creators of paradigm shifts. The authors in the collection, *Revolutions in Mathematics* [**Gil08a**], generally argue that a tenable notion of revolution in mathematics must be much more restrictive than Kuhn's.

[7]Model theorists loosely call a theory tame if it does not exhibit the *Gödel phenomena* – self-reference, undecidability, pairing function. One source of the word is Grothendieck's notion of 'tame topology'. See Chapter 5.6 for Wilkie's explanation and [**Tei97**] and [**dD99**].

(2) Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.
(3) The choice of vocabulary and logic appropriate to the particular topic are central to the success of a formalization. The technical developments of first order logic have been more important in other areas of modern mathematics than such developments for other logics.
(4) The study of geometry is not only the source of the idea of axiomatization and many of the fundamental concepts of model theory, but geometry itself plays a fundamental role in analyzing the models of tame theories.

At first glance the first thesis may seem banal. Isn't this just the justification for the study of symbolic logic? Isn't this claim just a rehash of positivistic themes of the 30's? Not at all. The examples illustrating the first aspect of Thesis 1, mathematical problems, concern specific[8] mathematical topics and we address them using modern model theoretic techniques (e.g. Chapter 5.6 and Theorem 9.3.3). For the second aspect one might ask, 'What is the philosophy of mathematics?" Avigad [**Avi07**] answers as follows, 'Traditionally, the two central questions for the philosophy of mathematics are: What are mathematical objects? How do we (or can we) have knowledge of them?' The traditional ontological issues are not in the scope of this book. Rather we address the second question by studying epistemological issues concerning the organization and understanding of mathematics as it is practiced. By a local foundation for a mathematical topic, we mean a specification of the area by a set of axioms. Hilbert in the *Grundlagen* and Bourbaki use informal axioms. A key goal[9] of this book is to show the mathematical advantages of stating these axioms in (usually first order) formal logic.

We consider reliability and clarity to be complementary objectives in the epistemology of mathematics. With Manders [**Man87**], we see clarity as at least as an important an issue as reliability[10]. Thus, we examine how the understanding of mathematical concepts changes (say the notion of number for the Greeks or for us now), and how formalization forces a clarifying analysis of concepts. For example, the first order axiomatization of geometry (Chapter 8-9 provides much finer information than the second order axiomatization.

One might incorrectly suspect the book is a defense of the 'formalism' leg of the foundational triumvirate. Rather, we deal with 'formalization' as a scheme for organizing mathematics without addressing any of the ontological concerns of 'formalists'.

The second thesis addresses the deep interactions between model theory and traditional mathematics. We describe how the new paradigm, by focusing attention on the content of particular fields rather than on a reduction to a global theory of all mathematics, provides connections across fields of mathematics that leads to mathematical advances (Chapter 5.6). This fact is base data for the study of mathematical practice. Thus the epistemological focus extends from reliability to more general concerns of clarity and coherence. Our approach is aligned with those

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[8]We indicate similarities and distinctions from the original foundational project on page xiv.

[9]The details of formalization are laid out in Chapter chform. In that chapter we consider the preformal mathematics as a set of concepts; in Chapter 8 we take Detlefsen's view of a dataset of accepted propositions. Our view certainly fits within the notion of the *hypothetical conception of mathematics* described in [**FG08**].

[10]We elaborate this discussion on page xviii.

grouped as *philosophy of mathematical practice* on the web page of the APMP[11]: 'Such approaches include the study of a wide variety of issues concerned with the way mathematics is done, evaluated, and applied, and in addition, or in connection therewith, with historical episodes or traditions, applications, educational problems, cognitive questions, etc.' These approaches are exemplified, though in many different ways, by such authors as Arana, Ehrlich, Hallett, McLarty, Maddy, Mancosu, Manders, Schlimm and Tappenden.

This movement[12] is more fully described in the introduction to *The Philosophy of Mathematical Practice* [**Man08b**] in which Mancosu makes the philosophical aims more precise. He notes that Benaceraff's 'rightly influential' articles set the guiding question as 'how, if there are abstract objects, could we have access to
{PMP} them'. Then he goes on to describe the positive goals of his book.

> The authors in this collection ... believe that the epistemology of mathematics has to be extended well beyond its present confines to address epistemological issues that have to do with fruitfulness, evidence, visualization, diagrammatic reasoning, understanding, explanation, which are orthogonal to the problem of access to 'abstract objects'[13].

In this spirit the goal in this book is to study not just the logical foundations of mathematics, but to understand the role of logic in contemporary mathematics. This discussion invokes not only some existing philosophical literature, but programmatic pronouncements by such authors as Bourbaki, Hilbert, Hrushovski, Kazhdan, Macintyre, Pillay, Shelah, Tarski, and Zilber that often have influenced, if not determined, mathematical practice. We introduce Shelah's methodological command: 'find dividing lines' on the next page; it recurs often in the text when illustrating the organization of model theory. In *The Statesman*, a dialog devoted in part to analyzing 'good definition', Plato advises to 'cut through the middle'. In Chapter 12 we develop the analogy between this dictum of Plato and Shelah's principle of seeking 'dividing lines' to understand the relations among mathematical theories.

The third thesis has several aspects. Both mathematical and philosophical questions may have different answers depending on the choice of logic. Chapter 7 expounds the vast variation in the amount of entanglement with axiomatic set theory among first, infinitary and second order logic. Metatheoretic investigation of first order logic gives finer information than second order logic about categoricity,

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[11] Association for the Philosophy of Mathematical Practice. `http://institucional.us.es/apmp/index_about.htm`

[12] The distinction between various approaches to the philosophy of mathematics is well known. Various terms have been used to make the distinction. The side I refer to as 'traditional philosophy of mathematics' is dubbed 'philosophy of Mathematics' (Harris, page 30 of [**Har15**]) or 'Foundations of Mathematics' (Simpson in clarifying his view on the Foundations of Mathematics Listserve)). While what I call 'philosophy of mathematical practice' becomes 'philosophy of mathematics' (Harris) or 'foundations of mathematics' (Simpson) and philosophy of 'real mathematics' for Corfield [**Cor03**]. Unlike Corfield, as a mathematician and model theorist for 40 odd years I regard model theory as 'real mathematics'.

[13] page 1-2 of [**Man08b**]

definability, and axiomatization. This finer information, particularly about definability, provides not only spectacular pure model theoretic results[14] but new tools for the study of traditional mathematics.

Shapiro's *Foundations without Foundationalism: A case for second order logic* inspired the title of this work. He writes[15], 'One of the main themes of this [Shapiro's] book is a thorough anti-foundationalism. ...The view under attack is the thesis that there is a unique best foundation of mathematics and a concomitant view that there is a unique best logic – one size fits all. We have gone to some lengths to identify inadequacies of first order logic, and we have shown how second order logic, with standard semantics, overcomes many of these shortcomings.' Indicating his alternative on page 29, he quotes Skolem's characterization of an 'opportunistic' view of foundations: to have a foundation which makes it possible to develop present day mathematics, and which is consistent so far as known yet. Shapiro proclaims, 'We might say that it is foundations without foundationalism'. So he still seeks reliability but without the high standard, maximally immune to rational doubt, and reductionist nature of traditional programs.

Shapiro then lays out a positive argument for founding mathematics using second order logic. He argues that basic analysis is comfortably axiomatized in second order logic and that basic notions such as closure of a subgroup, well-order, and infinite are all naturally defined in terms of second order logic. He identifies first order logic as deficient because it is 'subject to the compactness and Löwenheim theorem' (page 111).

The view here is somewhat orthogonal. Our position is not anti-foundationalist; we just choose to study other issues. On the one hand, we agree that one should study particular areas of mathematics rather than seeking a single foundation. And we agree that normal informal mathematical reasoning would most easily be formalized in second order logic. But in contrast to discussing the foundations of arithmetic and analysis, our focus is on the role of formalization in solving problems of modern mathematics. For us, compactness and categoricity in power for first order logic are powerful tools for understanding mathematics, not a deficit.

We could restate the fourth thesis as: geometry is the missing link that must be added to Bourbaki's three 'great mother-structures' (group, order, topology). (page 27) that are intended to organize mathematics. Geometry also unites what might appear to be disparate facets of this book. In studying the axiomatization of elementary geometry we highlight Hilbert's use of only first order axioms to prove a geometry admits a system of coordinates (as in high school geometry) over some field. The crucial property of geometry – a clear concept of dimension – is distilled in the notion of a *combinatorial geometry*[16]. This general notion applies equally well to finite as well as 'continuous' geometries. The stability classification of theories allows one to determine those theories T whose models admit combinatorial geometries. For those that do, it is possible to develop a structure theory for the models of T, where the building blocks of the models are geometries. The methods proving this result have many applications in traditional mathematics. Conversely,

[14] Much of this book is devoted to explaining the mathematical, philosophical and methodological significance of the *main gap theorem* (Theorem 5.5.3). It established a dichotomy between two kinds of theories; ones where the number of models in $\aleph_\alpha$ grows slowly with α and those where the number of models is the most it could be for every $\alpha > 0$. See page 29.

[15] Page 220 of [**Sha91**]

[16] See Chapter 5.4.

Hrushovski's field configuration, ultimately inspired by Hilbert's proof of the existence of a field, constructs classical groups and geometries from general model theoretic hypotheses with no algebraic or geometric hypotheses. Thus classical mathematics and model theory are inextricably intertwined.

After explaining in the next few paragraphs what I mean by the paradigm shift, I will try to clarify the purpose of this book by comparing approaches to the methodology of mathematics articulated by Franks, Maddy, Manders, and Tarski. Then I will move to a more detailed discussion of the contents of the book. I make reference in this discussion to many concepts of model theory and some deep mathematics. Familiarity with at least an upper-division undergraduate logic course is assumed. More advanced model theoretic notions are introduced, I hope gently, and indexed. Some details in the introduction are intended for those with more background; but, they should become clearer by examining the treatment in the text. There is little attempt to explain in any depth the concepts and results in other areas of mathematics. I do attempt to give a broad picture to show interactions both between areas and with model theory.

{parshift} **Features of the paradigm shift.** The paradigm shift that swept model theory in the 1970's really occurred in two stages. During the first stage in the 1950's and 1960's the focus switched from the study of properties of logics[17] to the study of particular (primarily first order) theories (the logical consequences of a set of axioms) and properties of theories and their impact on the models of the theories[18]. Robinson's identification of *model complete* theories is an early example of studying a *class* of theories; another is Morley's analysis of $\aleph_1$-*categorical* theories. The Ax-Kochen-Ershov proof of the Lang conjecture proceeds by identifying *complete* theories of Henselian valued fields[19].

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[17]Thus typical theorems involved such notions as decidability, interpolation; that is, assertions true for any theory. See Chapter 1.3 and 7.1.

{russellq} [18] Contrast this study of particular fields (theories) with the goals enunciated by Russell in the preface to [**RW10**]. 'We have however avoided both controversy and general philosophy and made our statements dogmatic in form. The justification for this is that the chief reason in favour of any theory on the principles of mathematics must always be inductive, i.e., it must lie in the fact that the theory in question enables us to deduce ordinary mathematics. In mathematics the greatest degree of self-evidence is usually not to be found quite at the beginning but at some later point; hence the early deductions, until they reach this point, give reasons rather for believing the premisses because true consequences follow from them, than for believing the consequences because they follow from the premises.' We agree with this analysis of the process of axiomatization. We argue however that experience has shown that the formulation of such premises for the principles of mathematics (in PM or from a contemporary perspective ZFC) stray too far from the actual practice of mathematics to properly illuminate it. Thus we pass from global to local foundations. In Chapter 3 we celebrate the success of second order axiomatization of certain canonical structures that meet Russell's criteria for a particular area of mathematics. Most of the text describes various first order theories whose axiomatizations are based on the same principle. We analyze below the limitations (for the case of Hilbert's geometry) of the description closing this same paragraph of Russell, 'All that is affirmed is that the ideas and axioms with which we start are sufficient, not that they are necessary. Our notion of 'modest descriptive axiomatization (Chapter 8) provides empirical criteria for describing overreach of an axiom set.

[19]Ax and Kochen won the 1967 American Mathematical Society Cole prize in number theory for their solution of Lang's conjecture that every polynomial of degree d with at least d^2 variables has a solution in the field of formal power series over the field with p elements. Ershov obtained
{Ax-Kochen} the result independently in the Soviet Union.

In the second stage, Shelah's decisive step was to move from merely identifying some fruitful properties (e.g. complete, model complete, $\aleph_1$-categorical) that might hold of a theory to a *systematic* classification of theories. As described in Chapter 5.3, he divides complete first order theories into four categories, each characterized by a syntactic property. The aim is to determine the class of theories whose models have a structure theory in a precise sense: each model is determined by a *system* of cardinal invariants. Shelah introduces the methodological precept of a 'dividing line' (Chapters 7.4, 12, and 2.4). He formulates each dividing line property so that theories that fall on one side (e.g. unstable) are creative; their models cannot be systematically analyzed as composed of small models, essentially new models are increased as the cardinality increases, and there are the maximal number of models. Models of theories on the other side (stable) have a 'structure theory' (i.e. they admit a local dimension theory). In the classifiable[20] case, each model is determined by a (well-founded) tree of cardinal invariants. This book investigates some consequences for mathematics and philosophy of mathematics of that paradigm change from: study the properties of logics (compactness, interpolation theorems, etc.) to: study virtuous properties of theories. A property of theory T is virtuous (Chapter 2.3) if it impacts the understanding of the models of T. A property is a 'dividing line' if both it and its complement are virtuous.

While the stability classification provides precise mathematically formulated dividing lines, 'tame vs. wild' is a less formal notion. The first order theory of arithmetic, $\mathrm{Th}(N, +, \cdot)$, was originally seen to be wild not only because it is essentially undecidable but because it admits a pairing function and so loses the essential geometric distinction between a structure A and the 'plane over it', A^2; thus there can be no notion of dimension. And so, there is no geometry on $(\mathbb{N}, +, \cdot)$. Since humans can only comprehend a small number of alterations of quantifiers, the existence of definable sets of arbitrary quantifier rank prevents a clear intuition[21] of the structure $(N, +, cdot)$. Shelah's taxonomy further shows that arithmetic has both of the strongest non-structure properties (the strict order property and the independence property[22]). Together these conditions help to explain why little of modern *algebraic* number theory takes place directly in first order Peano arithmetic (as opposed to logical analysis showing a particular result is provable in Peano). Rather, auxiliary more tame structures such as algebraically closed or valued fields provide the framework for proofs of number theoretic results[23].

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[20]A theory is classifiable if it satisfies each of the three dividing lines superstable, NDOP, NOTOP; see page 79 for these acronyms.

[21]By intuition, I mean the usual usage of mathematicians, a rough understanding of a concept or mathematical object and not any of the technical philosophical meanings. My intent is in the spirit of the first page of the Grundlagen [**Hil71**] where Hilbert writes, 'This problem [axiomatizing geometry] is tantamount to the logical analysis of our intuition of space'. Shapiro (page 39 of [**Sha91**] suggests 'tentative preformal beliefs'. Although one often thinks of the natural numbers as a clearly given structure, this assertion rests on a confusion. As Roman Kossak has pointed out, a clear intuition or vision of the natural numbers with successor is often confused with a clear {vocabintuit} intuition of arithmetic, the natural numbers with both addition and multiplication; few, if any, actually have the second intuition. See Chapter 1.2 for the role of the vocabulary and page 37 for further development of 'intuition of a structure.

[22]These are syntactical conditions, which each imply the maximal number of models in each power; technical definitions are on page 78.

[23]The epistemological significance of such a reduction is explored in [**Man87**].

We will explore the origins in the 1950's of this shift, its fruition in the 1970's, and the more mature pattern that developed in the 1980's. We consider some consequences of modern model theory for mathematics and for the philosophy of mathematics. Consequences for the latter arise in two ways: using the insights of modern model theory to develop existing research lines in philosophy (purity, categoricity, etc.); initiating the study of issues (e.g. the exceptional nature of $\aleph_0$, the role of model theory in organizing mathematics) in the philosophy of mathematics that are first seen from this new perspective. These issues all contribute to the emerging study of the philosophy of mathematical practice (e.g. [**Man08a**]).

The context of this work. Why should this change of mathematical goal from studying logics to studying the classification of theories have any impact on the study of the philosophy of mathematics? In contrast to the traditional program that studies the *Foundations of Mathematics* by constructing a single formal theory supporting all of mathematics, this shift empowers a strategy of defining many different formal theories to describe particular areas of mathematics.

Tarski's phrase, the methodology of deductive systems, is at the heart of the discussion. We proceed less ambitiously than Tarski,[24] whose *Introduction to Logic and the Methodology of the Deductive Sciences*, aimed

> to present to the educated layman . . . that powerful trend . . . modern logic . . . [which] seeks to create a common basis for the whole human knowledge.

Thus, unlike the analytic philosophy of the 1930's or even the work of Putnam and Montague mentioned in *the provocation*, there is no claim that the methods considered here are broadly applicable to the foundations of science. Rather than the broad program espoused by Tarski, our more modest goal is to expound for philosophers and mathematicians how the formal methods, initially springing from Tarski, Robinson, and Malcev but greatly extended in the wake of the paradigm shift, enhance the pursuit and organization of mathematics and the ability to address certain philosophical issues in mathematics.

Our approach is closer to that of two works: Maddy's *Second Philosophy: A Naturalistic method* [**Mad07**] and *Defending the Axioms* citeMaddydef. In the latter, she analyzes the methodology and the justification of the axioms of one particular first order theory, set theory. To see the analogy with the current book, we modify her[25] Second Philosopher's account of issues[26] arising in the study of set
{maddy2phil} theory, replacing each occurrence of 'set' in Maddy's text by 'model'.

> When our Second Philosopher is confronted with contemporary *model* theory, we've seen that questions of two types arise. The first group is methodological: What are the proper grounds on which to introduce *models*, to justify *model* -theoretic practice, to adopt *model*-theoretic axioms? The second group is more traditionally philosophical: what sort of activity is *model* theory? how does *model*-theoretic language function? what are *models* and how do we come to know about them?

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[24]Page xi of [**Tar65**].

[25]In [**Mad07**], Maddy begins to describe the method of inquiry of a *second philosopher*; by distinguishing it from the Cartesian method.

[26]See page 41 of *Defending the Axioms* [**Mad11**].

Some of these questions resonate immediately in their new context. Our discussion of formalization[27] in Chapter 1 considers the grounds for introducing models. On the other hand, we are not trying to formalize model theory, so the explicit question 'What are the proper grounds to adopt model-theoretic axioms?' is not in view. We seek rather to analyze the fundamental techniques and principles of model theory. Indeed, one might wonder whether a potential formalization is remotely possible. The 'grounds for adopting axioms' question raised by Maddy is one level too abstract for our study of model theory. Finding grounds for accepting the axioms is a task for a model theorist in formalizing any particular theory, including set theory, not about the field of model theory itself. As *the provocation* indicated, thinking of set theory in the same way as any other first order theory has proved its worth as a methodological standpoint in the last 50 years[28]. For us, the issue of intrinsic and extrinsic justification for axioms, a central concern for Maddy [**Mad11**], arises for any field of mathematics, from Euclid/Hilbert geometry (Chapter 8) to Hrushovski's contemporary theory of algebraically closed fields with an automorphism (Chapter 2.4).

The specific questions Maddy labels as 'traditionally philosophical' in the quotation at hand are among those this book intends to address. Much of the book describes 'the activity of model theory' and tries to explicate some of the directions of research. Thus, by *model theory* we almost always[29] refer to the study of the interaction between a collection of sentences in a formal language[30] and structures that satisfy those sentences. Much of our analysis concerns the function of model theoretic language. In seeking to understand how we come to know about models, we will study their properties and their relation with theories and classes of theories. While we focus on first order theories, the properties of models will be second order: e.g. prime, saturated, and universal.

Much of [**Mad11**] discusses the relation between various forms of realism and the choice of axioms for set theory. Our direction here is more in line with page 359 of [**Mad07**]: {Maddymethod}

> In sum, then, the Second Philosopher sees fit to adjudicate the methodological questions of mathematics – what makes for a good definition, an acceptable axiom, a dependable proof technique – by assessing the effectiveness of the method at issue as means towards the goal of the particular stretch of mathematics involved. Straightforward examination of the historical record suggests that theories about the nature of mathematical existence and truth don't play an instrumental role in these determinations, but this is not to say that such metaphysical questions evaporate completely from the second-philosophical point of view.

Here are some pertinent examples of methodological issues: Shelah's program of setting 'dividing lines' as normative assertions about the notion of 'good definition' (Chapter 12); the appropriate axioms for geometry (Chapter 8); and the nature of

[27]See page 5.

[28]See [**She99**] for Shelah's pragmatic approach to the choice of axioms for set theory.

[29]Chapter 13 allows more generality.

[30]Thus a 'formal theory', abbreviated in context to theory' reflects a mathematical theory in the usual informal sense such as field theory or matrix theory.

proof techniques in model theory (Chapter 4, 5 and 7). Two further methodological schemes are the role of test questions (Chapter 12) and the use of strong hypotheses (e.g. extensions of set theory) to obtain conclusions that one hopes to later establish with weaker hypotheses (Chapter 7.6). Thus our approach falls between those of Tarski and Maddy. Our scope is much narrower than Tarski's logic of deductive systems but also wider (from one standpoint) than Maddy's. The aim of Maddy's *Defending the Axioms* is to justify one first order theory for all of mathematics; in contrast, we are trying to understand what the goals of justification should be for different theories of various areas of mathematics. We approach global mathematical issues not by seeking a common foundation but by finding common themes and tools for various areas, not in terms of the topic studied, but *in terms of common combinatorial and geometrical features isolated by formalizations of each area.*

Another inspiration for this work is Franks' [**Fra10a**] study of the Hilbert program in *The Autonomy of Mathematical Knowledge.* In his introduction, Franks strikes a chord that will resonate in the current book: 'The first theme is that questions about mathematics that arise in philosophical reflection–questions about how and why its methods work–might be best addressed mathematically.' But then he continues:

> The second theme arises out of the first. Once one sees mathematics potentially providing its own foundations, one faces questions about the available ways for it to do so. The two most poignant issues are how a formal theory should refer to itself and how properties about a theory should be represented within that theory.

Here we part company. Since I seek no global theory of mathematics, there is no self-reference problem. Indeed, two key insights of modern model theory are that i) large amounts of modern mathematics can be better understood by formal systems which are tame (page x), so do *not* support self-reference, and ii) this tameness is actually constitutive of the fertility[31] of these theories in mathematical practice.

{reliabilitydis} As is common in model theory, we adopt a rather strong metatheory, ZFC[32]. However, we are interested (Chapter 7.6) in the possibility of weakening or strengthening model theoretic results within the general framework of axiomatic set theory. Such investigations can clarify the distinctions between making hypotheses about a specific topic and postulating general combinatorial principles in set theory. We do deny that the reduction of mathematics to set theory, designed for reliability purposes, is adequate for the understanding of mathematical practice. This is not to reject the question of justifying the ZFC axioms but to table it while discussing the role of formalization in clarifying mathematical discourse. Even so, we return to the reliability issues by noting in Remark 9.2.3 that Tarski's autonomous foundation for geometry is finitistically consistent so providing weaker meta-theory in one case.

Ken Manders [**Man87**] clarifies the contrast between the traditional focus on reliability and our focus on the clarity and interaction of mathematical concepts.

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[31]That is, we show (Chapter 2.4, 5.6) how the fact of the tameness can be exploited to prove mathematical theorems.

[32]Zermelo-Fraenkel set theory with the axiom of choice.

He outlines the distinction between the foundational[33] and the model theoretic approach. He begins colloquially[34] by stipulating that traditional epistemology concerns the correctness of mathematical assertions taken as knowledge claims.

> Is it **all right?**, traditional epistemology asks about knowledge claims. All schools in "logical foundations of mathematics" share this concern for reliability. But a long-term look at achievements in mathematics shows that *genuine mathematical accomplishment consists primarily in making* **clear** *by using new concepts*: ...Representations and methods from the reliability programs are not always appropriate. We need to be able to emphasize special features of a given mathematical area and its relationships to others, rather than how it fits into an absolutely general pattern. Model theoretic algebra works in just this way. A model theoretic approach may be able to bring out the point of algebraic methods in number theory and geometry.

Manders argues that a crucial aspect of mathematical progress is the introduction of new concepts to clarify a particular area. We illustrate this insight in various contexts, e.g., in our discussion of purity in Chapter 11. While in [**Man87**], Manders discusses only model theoretic algebra, we argue that not only the standpoint of model theoretic algebra (Chapter 4.4), but to an even greater degree the standpoint of the 'one model theory' (Chapter 5.7) that was obtained by integrating the methods of classification theory (Chapters 5.5 and 5.6) with model theoretic algebra, clarifies and unifies concepts in various areas of mathematics by finding unexpected similarities across fields of mathematics. Thus we don't abandon the epistemological enterprise but we focus on clarification rather than verification. How does one shape mathematical theories to best represent the inherent logic of the material? What similar patterns of reasoning or combinatorial features appear in various areas of mathematics?

The book is arranged as a web as well as a narrative. That is, we try to expound the basics of various model theoretic notions in terms of their methodological significance. A notion often has more than one such significance, so we have extensive cross-referencing in the text. A further goal is to attack the idea that after the foundational crises of the early 20th century mathematicians stopped engaging with philosophical issues. To that end, we frequently quote from expository articles, International Congress of Mathematics talks, and other sources in which mathematicians have laid out programs that not only raise specific mathematical problems but proclaim norms for 'good mathematics' and fruitful directions for research. We now summarize in more detail the contents of the book to fill out the description we have just given of the overall goals.

This book is *not* a text in model theory[35]. We do however sketch major ideas and results of model theory to illustrate our Theses 1-4. The exposition is organized

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[33]By this I mean a proposed global axiomatic system to include all (or almost all) of mathematics.

[34]The italics in the quotation are mine, but the boldface is original.

[35]We give some basic definitions, state some milestone results and give a feel for the methods involved in establishing them. But we rely on such expositions as the Stanford Encyclopedia of Philosophy for basic notions, [**Mar02**] and [**Hod87**] for more technical concepts, and refer the interested to advanced texts in stability theory for further details.

around methodological and historical themes. Thus, Shelah's theorem II.2.13 of [**She78**] reports nine equivalent definitions of a stable theory. These diverse statements are methodologically crucial. One of the key points of his 'dividing lines' program (Chapter 12.3) is that equivalent versions of the same definition play entirely different roles. We will in fact discuss three or four equivalents to stability – in different sections of the book – (Theorem 7.2.4, Theorem 7.4.2, Chapter 2.4).

We now describe the three parts of the book and connect them with our general theme.

1. Rethinking Categoricity. Michael Detlefsen raised a number of questions about the role of categoricity. The attempt to answer these questions has shaped {Detques} a good portion of this book.

> Question I[36]: (A) Which view is the more plausible—that theories are the better the more nearly they are categorical, or that theories are the better the more they give rise to significant non-isomorphic interpretations?
>
> (B) Is there a single answer to the preceding question? Or is it rather the case that categoricity is a virtue in some theories but not in others? If so, how do we tell these apart, and how to we justify the claim that categoricity is or would be a virtue just in the former?
>
> Question II: Given that categoricity can rarely be achieved, are there alternative conditions that are more widely achievable and that give at least a substantial part of the benefit that categoricity would? Can completeness be shown to be such a condition? If so, can we give a relatively precise statement and demonstration of the part of the value of categoricity that it preserves?

Further discussion revealed different understandings of some basic terminology. Does categoricity mean 'exactly one model', full stop? Or does it mean exactly one model in a given cardinal? Since Morley's ground breaking categoricity theorem [37] the actual meaning among model theorists for the colloquial 'categorical', is 'categorical in an uncountable cardinal'. In the usual first order model theoretic situation, the one model interpretation is trivial (it means finite). Is a theory automatically closed under (deductive/semantic) consequence? Is the topic 'theory' or 'axiomatization'?. Detlefsen's concerns were primarily about first or second order axiomatizations to provide descriptive completeness[38] for a particular area or for all of mathematics. In contrast, model theorists consider primarily (complete) first order theories. These different perspectives yield two roles for formalization in mathematics: as a foundational tool and as a device in the mathematician's toolbox.

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[36]Question I was questions III.A and III.B in a 2008 letter. Question II was question IV in the Detlefsen letter. I thank Professor Detlefsen for permission to quote this correspondence.

[37]Morley's theorem asserts that a first order theory is categorical in one uncountable cardinal κ (all models of that cardinality are isomorphic) if and only if it is categorical in all uncountable powers. Morley received the Steele prize from the American Mathematical Society for the seminal {catpower} influence of this work.

[38]In [**Det14**], Detlefsen distinguishes between 'descriptive completeness' and 'completeness for truth'. We address this notion at length in Chapter 8.1.