Philosophical implications of the paradigm shift in model theory

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The paradigm shift that swept model theory in the 1970's really occurred in two stages. During the first stage in the 1950's and 1960's the focus switched from the study of properties of logics to properties of theories. In the second stage, Shelah's decisive step was to move from merely identifying some fruitful properties (e.g. complete, model complete, $\aleph_1$-categorical) that might hold of a theory to a *systematic classification* of complete first order theories. Model theorists now undertake a systematic search for a finite set of syntactic conditions which divide first order theories into disjoint classes such that models of different theories in the same class have similar mathematical properties. With this framework one can compare different areas of mathematics by checking where theories formalizing them lie in the classification.

The following two theses express two aspects of the philosophy of mathematical practice. On the one hand model theory is studied as an example of mathematical practice; on the other, this analysis provides new perspectives in the epistemology of mathematics.

1. Contemporary model theory makes formalization of *specific mathematical areas* a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).

2. Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

Martin Davis wrote,

> Gödel showed us that the wild infinite could not really be separated from the tame mathematical world where most mathematicians may prefer to pitch their tents.

We will describe some aspects of the paradigm shift and show how it separates the tame world where most mathematicians pitch their tents from the wild world of arithmetic and set theory. This separation has allowed model theorists to better understand such central topics of modern mathematics as geometry (real, algebraic, and diophantine) and differential algebra. From the standpoint of the philosophy of mathematical practice the focus is changed from justifying the *reliability* of mathematical results to the *clear* understanding and organization of mathematical concepts as foreseen in [Man87].

Isolating the Stone space of a model of a theory T as a fundamental invariant for understanding T was a crucial step in the first stage of the shift. Grosholz [Gro85] and Schlimm [Sch85] recount successive

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analogies of propositional logic with algebra by Boole, of Boolean algebras with rings by Stone, and of deductive systems with Boolean algebra by Tarski that led to a remarkable unification of topology and logic. Thus the search for formal axioms creates mathematical advances. Lindenbaum and Tarski represent the space of completions of a first theory as the Stone space of its Boolean algebra of sentences. Then Vaught, Ryll-Nardjewski and Morley extend this analysis to describe the types (that is describe points in a model). They recognize the significance of the cardinality of these Stone spaces for determining properties of the set of models.

A second fundamental step is the development of the notion of universal domain. Manders [Man89] concept of domain extension clarifies the search. The choice of the right universal domain for a particular study (say in differential algebra) involves both what Manders calls context-external and context internal extensions. Model theory provides a uniform account of these extensions with wide applications in mathematics. The object of the extension is to construct an appropriately sized saturated model. But in [MV62], the cardinalities in which saturated models exist seem to be highly cardinal dependent and even connected to the continuum hypothesis.

Shelah's introduction of the stability hierarchy, the crux of the second stage of the paradigm shift, removes this cardinality problem by restricting it to the wild theories. A theory T is stable in λ if for every $M \models T$ with cardinality λ, $|S(M)| = \lambda$. Shelah proves that complete first order theories are divided into four classes depending on whether is T is stable in all λ (i.e. ω-stable), on all $\lambda \geq 2^{\aleph_0}$ (superstable), on a cofinal set of λ (stable) or never. Further if a theory is ever stable, it has a saturated model in exactly those cardinals in which it is stable and it admits (at least locally) a dimension theory. Since (non-trivially), algebraically closed fields, differentially closed fields, and compact complex manifolds are all ω-stable they share mathematical properties which both suggest analogous theorems or more strongly allow metatheorems that apply in different areas. Now formalization is a tool for organizing contemporary mathematics. Unlike the foundationalist school, this framework produces results in the mathematical fields studied.

References

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