# Is ‘structure’ a clear notion?

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago*

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The following message was recently posted on the FOM listserve

Robin Adams robin.adams78@gmail.com via cs.nyu.edu Feb 15 (2 days ago) Reply to Foundations At the same time as this discussion, there is a discussion on the category theory mailing list about the definition of ”structure”. Jean Benabou pointed out that the word is used often, and in a way that suggests there is a definition on which all mathematicians agree, but this is misleading. There have been a few attempts (e.g. in Bourbaki), but none has been generally accepted by the whole community of mathematicians.

A ”forgetful functor” is a functor from a category whose objects have ”more structure”, to a category whose objects have ”less structure”. We know intuitively what we mean when we say that a group is monoid with some ”additional structure”. We can write ”Let U be the forgetful functor from the category of groups to the category of monoids”, and everyone knows what we mean. But a formal definition is elusive.

Very like the case of ”definite property” in Dedekind’s set theory as opposed to Zermelo’s - everyone agrees on the individual cases, but there is as yet no precise general definition.

So, in my opinion, ”structure” and ”forgetful functor” are vague concepts in today’s mathematics.

*end quote*

In contrast, the following extract from my book give a clear notion of a $\tau$-structure for a fixed vocabulary $\tau$ and explains why the $\tau$ is essential.

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# 1 Formalization

Suppose we want to clarify the fundamental notions and methods of an area of mathematics and choose to formalize the topic. Our notion of a formalization of a mathematical topic[1] involves not only the usual components of a formal system, specification of ground vocabulary, well-formed formulas, and proof but also a semantics. From a model theoretic standpoint the semantic aspect has priority over the proof aspect. The topic could be all mathematics via e.g. a set theoretic formalization. But our interest is more in the local foundations of, say, plane geometry or differential fields. We set the stage for developing Thesis 1, by focusing on a specific vocabulary, designed for the topic[2] rather than a global framework. In any case, a mathematical topic is a collection of concepts and the relations between them. There are course other less restrictive notions of axiomatization, and often such a 'formalism' deliberately omits the semantic aspect. But we want the wider notion here as it reflects the model theoretic perspective.

It is not accidental that 'formalization' rather than 'formal system' is being defined. The relation between intuitive conceptions about some area of mathematics (geometry, arithmetic, Diophantine equations, set theory) and a formal system describing this area is central to our concerns. The first step in a formalization is to list the intuitive concepts which are the subject of the formalization. The second is to list the key relations the investigator finds among them. In stipulating this view of formalization, we are not claiming to fix the only meaning of the term but only the meaning most suitable for the discussion here.

**Definition 1.0.1.** *A* full formalization *involves the following components.*

1. *Vocabulary: specification of primitive notions.*
2. *Logic:*
   (a) *Specify a class*[3] *of well formed formulas.*
   (b) *Specify truth of a formula from this class in a structure.*
   (c) *Specify the notion of a formal deduction for these sentences*[4].
3. *Axioms: specify the basic properties of the situation in question by sentences of the logic.*

We think of the specification in Definition 1.0.1 as given in informal set theory; the usual mathematicians' 'could formalize in ZFC' applies. In the critique of

[1]In general, context, area, and topic are synonyms in this book. See Chapter **??**.

[2]Thus, propositional logic and its variants, which analyze general reasoning as opposed to specific mathematical content, are not discussed here.

[3]Most logics have only a set of formulas, but some infinitary languages have a proper class of formulas. While we consider only classical logics, the framework doesn't make this restriction.

[4]Such a notion will generally involve 'logical axioms'; when we speak of axioms we generally mean those in item 3. Euclid more precisely calls them *postulates* but such a distinction now seems pedantic. We sometimes discuss logics that have no deduction system.

some problems from a calculus text (Example 2.0.4) we will see how crucial item 1 is to avoid confusions in a situation where there is no thought of a formal language as described in 2a). An even more basic example of the importance of fixing vocabulary is the confusion in elementary algebra between whether the minus sign represents binary subtraction or the unary operation of taking the additive inverse.

# 2 Vocabulary and Structures

We establish some specific notations which emphasize some distinctions between the mindsets of logicians, in particular, model theorists, and 'normal' mathematicians. Indeed, this insistence on a vocabulary relevant to the content being studied is one aspect of the paradigm shift and we study its emergence as a key precept of model theory in Chapter **??**.

**Definition 2.0.2.** *1. A* vocabulary $\tau$ *is a list of function, constant and relation symbols.*

2. *A* $\tau$-structure[5] $\mathcal{A} = \langle A, R_1, \dots R_n\rangle$ *is a set* $A$ *(the domain of* $\mathcal{A}$*) with an interpretation of each symbol in* $\tau$*. That is, for each* $n$*-ary relation*[6] *symbol* $R$ *in* $\tau$*,* $R^A$ *is a collection of* $n$*-tuples from* $A$*.*
3. *A structure*[7] *is* many-sorted *if there are a family of unary predicates* $T_i$ *and each the variable, function, and relation symbols assign sorts to their arguments and, in the case of functions, values.*

We employ in *this chapter* a useful, but in general archaic convention, of writing $\mathbb{N}$ for the domain, function and relations and $N$ for the domain of a structure. Thereafter, we use current model theoretic notation and write $N$ for both, as there is usually little chance of confusion. In that case $|N|$ denotes the cardinality of the domain, as $|X|$ always denotes the cardinality of a set $X$.

Specifying a vocabulary (signature, similarity type)[8] is only one aspect of the

[5] Philosophers frequently use the word system (Shapiro [1997], 92).

[6] Do the same for constant and function symbols.

[7] There are some minor technical issues around many-sorted structures for the development of $T^{eq}$ that are discussed in Chapter **??**.

[8] The actual general definition of an abstract algebra was made by Garrett Birkhoff in Birkhoff, Garrett [1935] ('As no vocabulary suitable for this purpose is current.') Moreover he explicitly introduces the idea of considering all algebras of a fixed *species* (vocabulary) and proves his famous theorem that a class of algebras is defined by a set of equations if and only if it is closed under homomorphism, subalgebra and product. Years later Tarski extended this notion to include relation symbols and used *similarity type* for essentially this notion; sometimes it is called the *signature*. These notions are implicit in Tarski [1950], Robinson [1952] Tarski [1954] and Tarski and Vaught [1956]: the concept of two systems having the same type is defined there rather than the emphasis I have placed on choosing a vocabulary as part of formalization. These notions (a specification of a sequence of arities) are one level of abstraction higher than vocabulary. But they have the same effect in distinguishing the syntactic from the semantic and we will use vocabulary. In 1953 Tarski et al. [1968] Tarski just specifies predicate and function symbols of prescribed arity. But

notion of a formal system. But it is a crucial one[9] and one that is often overlooked by non-logicians. From the standpoint of formalization, fixing the vocabulary is a first step, singling out the 'primitive concepts'. Considerable reflection from both mathematical and philosophical standpoints may be involved in the choice. For example, suppose one wants to study 'Napoleon's theorem' that the lines joining the midpoints of any quadrilateral form a parallelogram. At first sight, one might think the key notions (and therefore primitive concepts) are quadrilateral and parallelogram. But experience[10] even before Euclid showed that the central basic notions for studying the properties of quadrilaterals are point, line, incidence, and parallel[11] and the delineation of types of quadrilaterals is by explicit definition.

The choice of primitive notions for a topic is by no means unique. For example, formulated in a vocabulary with only a binary function symbol, the theory of groups needs $\forall\exists$-axioms (page **??**) and groups are not closed under subalgebra. Adding a constant for the identity and a unary function for inverse, turns groups into a universally ($\forall$: only universal quantifiers) axiomatized class that is closed under substructure. Alternatively, groups can be formulated with one ternary relation as the only symbol in the vocabulary. The three resulting theories are pairwise bi-interpretable.

The notion of isomorphism is often abused[12]; the notion only makes sense with respect to a specified vocabulary.

**Definition 2.0.3.** *We say two $\tau$-structures $A$ and $B$ are isomorphic ($A \approx B$) if there is a bijection $f$ between their domains such that for each $\tau$-relation (analogously for function, constant) symbol $R$, $R^A(\boldsymbol{a})$ if and only if $R^B(f(\boldsymbol{a}))$.*

Thus we might construct a structure $(N, S)$ in different ways; we could take the universe as the finite Von Neumann or Zermelo ordinals[13]. In the vocabulary $\{S\}$, the structures are isomorphic; in the vocabulary $\{S, \epsilon\}$, they are not. We explore this distinction further in Chapter **??**.

It is a commonplace in model theory that just specifying a vocabulary means little. For example in the vocabulary with a single binary relation, I can elect to for-

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Henkin Henkin [1953] is completely modern. Still another 'synonym' is language. We explain on page **??** why we try to avoid this word. Although Malcev opens his 1936 paper Malcev [1971] by specifying a finite set of functions (predicates) defined on a set $M$, he doesn't use (per the index of his collected works) the word (similarity) type or signature until around 1960.

[9]Demopoulos Demopoulos [1994] describes this issue as the role of non-logical constants in an insightful article on the Frege-Hilbert correspondence.

[10]Experience includes determining which definitions and postulates best conform to intuition and which allow a smooth development of the subject. Well-known precursors of Euclid include Eudoxus, Theudius, and Hermotimus (page 116 of Euclid [1956]).

[11]I note that alternatives basing geometry on regions rather than point/line for epistemological reasons were advanced by e.g. Whitehead and Leśniewski, but that is not relevant to this point.

[12]Structure could be defined purely axiomatically. A category is defined as a collection of 'objects' and 'arrows' (or morphisms); objects $A$ and $B$ are 'isomorphic' if there is a pair of inverse morphisms $f, g$ between them. That is, $f \circ g = 1_A$ and $g \circ f = 1_B$. This notion is useful in abstract category theory but most applications are to *concrete categories* of structures in the set theoretic sense.

[13]In the first case the successor of $\emptyset$ is $\{\emptyset, \{\emptyset\}\}$ while in the second it is $\{\emptyset\}$.

malize either linear order or successor (by axioms asserting the relation is the graph of a unary function). Thus, while I here focus on the choice of relation symbols – their names mean nothing; the older usage of signature or similarity type might be more neutral. The actual collection of structures under consideration is determined in a formal theory – by sentences in the logic. In the formalism-free approaches discussed in Chapter **??** the specification is in normal mathematical language. Having fixed a vocabulary with one binary relation, we say, e.g., 'Let $\boldsymbol{K}$ be the class of well-orderings of order-type $< \lambda$ such that ....'

But while axioms are necessary to determine the meanings of the relations in a vocabulary; the mere specification of the vocabulary provides important information. David Pierce Pierce [2011] has pointed out the following example of mathematicians' lack of attention to vocabulary specification.

**Example 2.0.4** (Pierce)**.** Spivak's Calculus book is, one of the most highly regarded texts in late 20th-century United States. It is more rigorous than the usual Calculus I textbooks. Problems 9-11 on page 30 of Spivak [1980] ask the students to prove the following are equivalent conditions on $N$, the set of natural numbers. This assertion is made without specifying the vocabulary that is intended for a structure $\mathbb{N}$ with domain $N$. In fact, $N$ is described as the counting numbers,

$$1, 2, 3, \ldots$$

**1) induction** ($1 \in X$ and $k \in X$ implies $k+1 \in X$) implies $X = N$.

**2) well-ordered** Every non-empty subset has a least element.

**3) strong induction** ($1 \in X$ and for every $m < k$, $m \in X$ implies $k \in X$) implies $X = N$.

As Pierce points out, this doesn't make sense: 1) is a property of a unary algebra[14]; 2) is a property of ordered sets[15] (and doesn't imply the others even as ordered unary algebras[16]); 3) is a property of ordered unary algebras. In particular, 2) is satisfied by any well-ordered set while the intent is that the model should have order type $\omega$.

It is instructive to consider what proof might be intended for 1) implies 3). Here is one possibility. Let $X$ be a non-empty subset of $N$. Since every element of $N$ is a successor (Look at the list!), the least element not in $X$ must be $k+1$ for some $k \in X$. But the existence of such a $k$ contradicts property 1). There are two problems with this 'proof'. The first problem is that there is no linear order mentioned in the formulation of 1). The second is, 'what does it mean to 'look at the list'?' These objections can be addressed. Assuming that $\mathbb{N}$ has a discrete linear order satisfying

[14]The vocabulary contains only the unary function $S$.
[15]The vocabulary contains only the binary function $<$.
[16]The vocabulary contains $<, S$.

$(\forall x)(\forall y)[x \leq y \vee y+1 \leq x]$ and that the least element is the only element which is not a successor resolves the first problem. This assertion follows informally (semantically) if one reads 'look at the list' as 'consider the natural numbers as a subset of the linearly ordered field of reals'.

As Pierce notes, a fundamental difficulty in Spivak's treatment is the failure to distinguish between the truth of each of these properties on the appropriate expansion of $(N, S)$ and a purported equivalence of the properties– *an equivalence which can make sense only if the properties are expressed in the same vocabulary*.

But in another sense the problem is the distinction between Hilbert's axiomatic approach and the more naturalistic approach of Frege. I'll call Pierce's characterization of Spivak's situation, Pierce's paradox. It will recur[17]; Pierce writes:

> Considered as axioms in the sense of Hilbert, the properties are not meaningfully described as equivalent. But if the properties are to be understood just as properties of the numbers that we grew up counting, then it is also meaningless to say that the properties are equivalent: they are just properties of those numbers. (Pierce [2011])

Note that this distinction about vocabulary is prior to distinctions between first and second order logic. We stated the difficulty in the purported equivalence of 1) and 2) in terms of second order logic. But the same anomaly would arise if Peano Arithmetic (with a schema of first order induction) were compared with 'every *definable* set has a least element'.

Pierce's paradox is fundamentally a semantic remark. Two sentences are equivalent if they have the same models; this makes no sense if they do not have same vocabularies or at least are viewed as sentences for a vocabulary that contains the symbols from each sentence. It might have been more precise for Pierce to say 'trivially false' rather than meaningless. In the second and third cases enumerated by Pierce, it is clear that as sentences in the vocabulary with symbols $(S, <)$ they are simply not equivalent. And trivially they are both true in the structure $(N; S, <)$. It can be objected that it makes sense to prove one property of a given structure $\mathcal{A}$ implies the truth of another on $\mathcal{A}$ using properties of $\mathcal{A}$. That seems a normal enough mathematical strategy. But consider the case at hand: on $(N, <, S)$, well-order implies induction (i.e. order type $\omega$). Why? Because it is a property of $N$ that the order type is $\omega$. But this seems to me to be just the type of argument I attribute to Spivak a few paragraphs up; it is hard to find a non-trivial phrasing of it.

We introduced vocabulary in Definition 1.0.1 as 'the specification of primitive notions'. Thus the choice of the vocabulary is the fundamental step in the formalization process. The vocabulary should focus attention on the concepts seen as most basic[18].

[17] See, in particular, just after Example **??**.

[18] We don't attempt to analyze the meaning of 'basic' or 'natural' concept; we just rely on the usual

For example, in algebraic geometry, the crucial objects (solutions of systems of equations) are represented by conjunctions of systems of equations (perhaps of high degree in several variables), that is conjunctions of atomic formulas in the vocabulary.

The following question is of real methodological importance; the two answers below are used in different contexts and with different results. What is the appropriate vocabulary and logic to study vector spaces over the reals?

**Example 2.0.5** (Formalizing modules)**.** Module is a generalization of vector space obtained by replacing the field of scalars by a ring. The standard formalization is as a first order single sorted theory in a vocabulary with a symbol $m_r$ for each $r \in \Re$ and $(+, 0)$. The axioms specify that single sort is an abelian group and each unary functions $m_r$ behaves as the scalar multiplication by $r$ on the vector space. The models of this theory are all real vector spaces.

An alternative is a first order *many-sorted logic*[19]. One can approach real vector spaces in a 2-sorted logic with a sort $F$ for field elements, a sort $V$ for vectors with field operations on $F$, group operations on $V$ and scalar multiplication from $F \times V$ to $V$. The models of this theory are all pairs of a real closed field $F$ and a vector space over $F$. If the intent is to study *real vector spaces* something must be done to restrict the $F$ sort. For instance, one can add an $L_{(2^\omega)^+,\omega}$ axiom (Definition **??**) insisting that every element of $F$ is a unique realization of a cut in the subfield $\mathbb{Q}$ (Each element of $\mathbb{Q}$ is definable over the empty set.).

For different purposes each of these is a plausible approach. But since the 1970's at least, the first approach is almost universally adapted. Without some infinitary restriction on the real sort[20]; the many-sorted theory is unstable (Definition **??**); while, the single-sorted version is categorical in all powers.

An important notion that depends on the precise understanding of vocabulary is that of a *pseudo-elementary class*. A pseudo-elementary[21] class in a vocabulary $\tau$ is the collection of reducts (forget the predicates and functions in $\tau_1 - \tau$) of models of a theory in a larger vocabulary $\tau_1$. In earlier days, a universal vocabulary was often assumed, generally, containing infinitely many $n$-ary relations for each $n$. In contrast, we seek primitive terms which pick out the most basic concepts of the field in question and axioms which in Hilbert's sense give us an implicit definition of the area. Thus, we can formalize concepts such as real closed fields (RCF) or algebraic geometry[22] without reference to the construction in set theory of specific models[23]. Our treatment

understandings in mathematical practice. A more detailed analysis would distinguish between the criteria of 'reflecting intuition' and 'provides a clear framework'.

[19]The universe is a disjoint union of sorts (i.e. unary predicates). The relation and function symbols specify to which sorts they apply. By fiat, if there are infinitely many sorts one can ignore elements that are not in one of the given sorts. See Definition **??**.

[20]For instance, one could write $\forall x[R(x) \to \bigvee_{r\in\Re} x = r]$; but the advantages of first order logic are lost

[21]Such a class is called $PC$ if the $\tau_1$ class is defined by a single sentence or in general $PC_\delta$ and $PC_\delta$ in $L_{\omega_1,\omega}$ if the sentences are in $L_{\omega_1,\omega}(\tau_1)$.

[22]See the discussion of Zariski geometries in Chapter **??**.

[23]Geometry and analysis are presented in this way in e.g. Spivak [1980], Hilbert [1971], and Heyting

of the primitive terms is analogous to the treatment of the element relation in set theory. This analysis is relevant to either traditional (global) or local foundations. For any particular area of mathematics, one can lay out the primitive concepts involved and choose a logic appropriate for expressing the important concepts and results in the field. While in the last quarter century model theory has primarily focused on first order logic as the tool, we discuss some alternatives in Chapters **??** and **??**.

Still another example of the subtlety of choosing primitive terms is given in Manders [1984]. Manders points out that the mutual interpretability between classical geometries and fields can only be treated as a transformation preserving model completeness by a very careful choice of the primitives for the geometry (particularly for geometries with an order on each line). He raises the general philosophical issue of obtaining a well-adapted logic for modern (i.e. scheme theoretic) algebraic geometry. As he put it (page 328), 'Why must even innovative attempts to use Tarski semantics, say with unobvious but geometrically intrinsic primitives, break down in describing modern algebraic geometry?' The stress on modern here is essential; in Example **??** we discuss the naturality of the model theoretic formulation of affine schemes *over fields* and thus for Weil-style algebraic geometry; working over arbitrary rings in Grothendieck style is a different ball game.

Our entire discussion concerns what are sometimes called 'first-order structures', which some distinguish from 'second-order structures'. I find such notions[24] of '$nth$-order structure' often conflate two distinct notions: a) structure and b) the semantics of a logic.

The notion of many-sorted (Chapter **??**) structure allows one to treat uniformly structures of any (finite for simplicity) order. There is one sort symbol in the vocabulary for each order; there may be predicate symbols in the vocabulary relating elements from the same or different sorts.

A language (in the sense of Shapiro [1991]) or a specification of well-formed formulas ((ii)a of Definition 1.0.1) or a grammar (Manzano [1996],6) describes the expressions of a logic based on such a vocabulary. There are countably many $k$-ary relation variables for each sort and for each $k$. As Section 3.3 of Shapiro points out there can be both first order and higher order semantics for this language. But confusion is introduced when an unnecessary notion of a standard 'Higher order structure[25]' appears. The underlying structure is no different – the scope of the quantifiers has changed. Valuable information is obtained by varying this scope as in the treatment of second order arithmetic Simpson [2009], where one might quantify over the recursive sets or the $\pi^1_1$ sets.

[1963].

[24]Corry's discussion Corry [1992] of Bourbaki's awkward and unused attempt to provide such a formalization exemplifies the resulting confusion. Strangely, Corry makes no mention of the standard notion of structure discussed here.

[25]A clear definition of this notion is given when (Manzano [1996],22) requires that 'all subsets of X' be the actual power set of $X$ (in the ambient model of set theory).

Abraham Robinson's theory of *nonstandard analysis* involves not only a nonstandard (and importantly saturated) model of the real field but is also able to deal with higher order objects Henson and Keisler [1986]. Because of its different techniques and formalization we do not include this important field in our analysis in this book.

The notion of fixing a vocabulary to study a family of structures is basic in modern model theory. But this convention is part of the paradigm shift. The study of logics in the early 1950's (e.g. Church [1956]) speaks of the first order predicate calculus which has function and relation variables (which can't be bound) and might have function constants as well. Even the founding papers of model theory are ambiguous about this point as discussed in Footnote 8. But the modern notion is fixed, although Chang and Keisler [1973] and Shoenfield [1967] use 'language' rather than 'vocabulary'.

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