Research Statement

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My primary research interest is the geometry and topology of manifolds of dimension 3. The Geometrization Conjecture of W. Thurston asserts that any 3-manifold may be cut in a canonical way along spheres and tori into pieces, each of which admits a Riemannian metric modeled on one of 8 homogeneous spaces, that is, simply connected manifolds with a transitive Lie group action. The study of 3-manifolds has been revolutionized in the past few years by work of G. Perelman on the Ricci flow with surgery introduced by R. Hamilton, and it appears that these techniques may be used to give a complete proof of Thurston's conjecture. However, the topology and geometry of 3-manifolds is by no means completely understood. Indeed, the *hyperbolic* 3-manifolds (admitting a metric of constant sectional curvature -1) are the richest and least understood class of 3-manifolds, and according to geometrization these are generic in the following strong sense. If the fundamental group of a closed 3-manifold is infinite, noncyclic, freely indecomposable, and contains no noncyclic abelian subgroups then the 3-manifold admits a hyperbolic metric. Geometric flows seem promising as avenues of approach to understanding the geometry even of manifolds already known to admit a hyperbolic structure, and below I will outline work using flows to understand negatively curved metrics naturally arising from Dehn surgery on hyperbolic 3-manifolds. I am also interested in algebraic and geometric questions about hyperbolic knot complements, and I will discuss my work in that area in the following section. More broadly, I am interested in hyperbolic manifolds of arbitrary dimension and character varieties, and in the final section I will sketch ideas in these areas.

Geometric flows and hyperbolic 3-manifolds

Given a metric g_0 on a 3-manifold M, the Ricci flow gives a family of metrics $g(t)$ on M with initial value $g(0) = g_0$, determined by the following partial differential equation:

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}(g)$$

where $\mathrm{Ric}(g)$ is the Ricci tensor of the metric g. This was introduced by R. Hamilton, who proved that if the initial metric has positive–definite Ricci tensor, then the flow converges in finite time to a metric of constant positive curvature after a suitable normalization [21]. For any initial metric there is some $T > 0$ and a smooth family of metrics $g(t)$ satisfying the Ricci flow equation for $0 \leq t < T$, but in general the flow may form singularities in finite time. The Ricci flow with surgery program attempts to describe these singularities in sufficient detail that one may resolve them by stopping the flow when (or shortly before) it comes to a singularity and altering the manifold ("surgery") so that when the flow is restarted the given singularity no longer occurs. Perelman's work ([38], [40], [39]) allows the surgery program to be used to prove the geometrization conjecture (see also [27], [8], [35]).

Agol-Storm-Thurston have used the monotonicity of a curvature–adjusted volume along the Ricci flow with surgery to give lower bounds on volumes of closed hyperbolic Haken 3-manifolds [4]—those which contain an embedded surface whose inclusion induces an injection at the level of fundamental group. This theorem is remarkable in that it uses the Ricci flow to give new information about manifolds already known (by work of Thurston, [46], [47], [48]) to admit a hyperbolic metric. In general, though, Ricci flow is less than ideal for manifolds of negative sectional curvature, as it does not preserve negative curvature and one may encounter infinitely many surgeries before convergence to hyperbolic.

Chow-Hamilton recently introduced the *cross curvature flow* [11], defined by the equation

$$\frac{\partial}{\partial t} g = -2X(g),$$

where $X(g)$ is the cross curvature tensor of g, defined as follows. Let $E_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ be the Einstein tensor, and raise indices in the usual way, $E^{ij} = g^{ia} g^{jb} E_{ab}$. The cross curvature tensor is $X_{ij} = \frac{1}{2} E^{uv} R_{iuvj}$, where R_{iuvj} is the Riemann curvature tensor.

Chow-Hamilton conjecture that the cross curvature flow is an analog for negatively curved manifolds of the Ricci flow on manifolds of positive Ricci curvature.

Conjecture (Chow-Hamilton, [11]). *Given an initial metric g_0 with negative sectional curvatures on a closed 3-manifold, the cross curvature flow exists for all time, preserves negative curvature, and, suitably normalized, converges to a hyperbolic metric as $t \to \infty$.*

According to geometrization, any 3-manifold which admits a negatively curved metric actually admits a hyperbolic metric. The conjecture of Chow-Hamilton would answer the following more precise question in the affirmative.

Question. *For a 3-manifold M admitting a metric of negative sectional curvature, does the space of all such metrics deformation retract to the hyperbolic metric?*

Various generalizations of this question are known to be false for manifolds of dimension greater than 3. Gromov-Thurston have given examples of negatively curved 4-manifolds which do not admit hyperbolic metrics [20], and even among those which do admit hyperbolic metrics, recent examples of Farrell-Ontaneda [16] in dimension 6 and higher show that the space of negatively curved metrics may not be contractible. Indeed, in general the space of negatively curved metrics on hyperbolic manifolds of dimension greater than 10 is not even connected [17]. According to the conjecture of Chow-Hamilton, in dimension 3 the (normalized) cross curvature flow should exhibit a deformation retraction from the space of negatively curved metrics to the hyperbolic metric.

Chow-Hamilton gave evidence for their conjecture by showing that a certain integral measure of the difference of a negatively curved metric from hyperbolic is monotone decreasing under the cross curvature flow, for as long as the flow exists. Dan Knopf and Andrea Young have given additional supporting evidence, showing that the cross curvature flow is stable at a hyperbolic metric [28]. It is easy to see that a hyperbolic metric evolves by scaling under cross curvature flow, and hence is fixed after normalization. The content of the theorem of Knopf-Young is that on a closed manifold admitting a hyperbolic metric there is some neighborhood U of the hyperbolic metric in the space of metrics modulo scaling, such that the conjecture holds for initial metrics $g_0 \in U$.

A natural test case for the conjecture of Chow-Hamilton arises from the "2π theorem" of Gromov-Thurston ([20], see also [6]). This displays a metric of negative curvature on the manifold resulting from Dehn surgery on a cusp of a finite-volume noncompact hyperbolic 3-manifold, provided that the surgery slope has geodesic length greater than 2π in the Euclidean metric on some cusp cross-section. According to geometrization, the resulting manifold actually admits a hyperbolic metric. It has proven fruitful in many contexts to compare the hyperbolic metrics before and after surgery, for example, Agol-Dunfield use volume inequalities of Agol-Storm-Thurston to give new lower bounds on the minimal volume of a closed hyperbolic 3-manifold [4]. Hodgson-Kerckhoff showed that there is a path in the space of incomplete metrics of constant sectional curvature between a finite-volume hyperbolic structure and certain of its surgeries [22]. Analogously, the cross curvature flow conjecturally defines a path between the "2π metric" and the hyperbolic metric in the space of negatively curved metrics.

With Knopf and Young, I am analyzing the cross curvature flow applied to rotationally symmetric diagonal metrics of negative curvature on the solid torus. These metrics are of the kind constructed on the solid torus in the proof of the 2π theorem, and their high degree of symmetry makes evolution equations for curvature quantities somewhat more tractable than the general case. Prescribing appropriate boundary conditions, we prove short–time existence and show

Theorem 1 (DeBlois-Knopf-Young, [15]). *The cross curvature flow on the solid torus preserves negative curvature.*

The fact that negative curvature is preserved is not generally known for cross curvature flow and is thus an important validation of one aspect of Chow-Hamilton's conjecture in this case. In fact we give bounds on the curvatures above and below for all time, and we have recently been able to give bounds on the derivatives of curvature as well. This should be enough to show long–time existence, after which it only remains to show convergence to hyperbolic to establish the full conjecture of Chow–Hamilton in this special case. I intend to study this further, as well as the general case of the conjecture and the specific setting of the 2π theorem.

The algebra and geometry of knot complements

Many interesting questions about the geometry of hyperbolic manifolds involve the algebraic invariants associated to them. The isometry group of hyperbolic 3-space is isomorphic to $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\}$, and a complete hyperbolic structure on a 3-manifold M therefore gives rise to a faithful representation of $\pi_1(M)$ onto a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{C})$. If M has finite volume, it is a consequence of the rigidity theorems of Mostow and Prasad that Γ may be conjugated into $\mathrm{PSL}_2(k)$, where k is some finite extension of $\mathbb{Q}$. A related object is the *trace field* of Γ, defined to be the extension $\mathbb{Q}(\{tr\,\gamma \,|\, \gamma \in \Gamma\})$. Evidently, the trace field is a subfield of the field containing the entries of Γ. In fact, it can be arranged by conjugating Γ to be a subfield of degree at most 2. Since trace is invariant under conjugation, the trace field is an invariant not just of Γ but of M (see for example [32] for an exposition). My advisor, Alan Reid, has asked the following question.

Question (Reid)**.** *Do there exist infinitely many hyperbolic knot complements with trace field of bounded degree?*

If two link complements in S^3 are *commensurable*—that is, if they share a finite cover—they have the same trace field. An negative answer to the question above would therefore imply the following conjecture of Reid.

Conjecture (Reid, [42])**.** *A hyperbolic knot complement has at most finitely many others in its commensurability class.*

There seems to be good evidence for this conjecture; for instance, it is known to be true for hyperbolic 2-bridge knot complements [41] and hyperbolic knot complements which do not cover an orbifold with a rigid cusp [36]. Furthermore, it follows from the cyclic surgery theorem [12] that a hyperbolic knot complement is covered by only finitely many others. In contrast, it is easy to construct infinitely many commensurable 2-component links. Indeed, if one component of a hyperbolic link L is unknotted and the other component links it once, then cyclic covers over the meridian of the unknotted component are 2-component links as well.

In joint work with Eric Chesebro I prove a theorem for links using techniques which may more directly generalize to the knot complement case.

Theorem 2 (Chesebro-DeBlois, [10])**.** *For any $k \geq 2$, there exist infinitely many commensurability classes of k-component links whose complements have trace field $\mathbb{Q}(i, \sqrt{2})$.*

As mentioned above, the accomplishment of this result lies in constructing the link complements to be *incommensurable.* These link complements are all obtained from two building blocks, one of which is a 2-string tangle S in the ball B^3 and the other of which is a 4-string tangle T in $S^2 \times I$. There is a well-known hyperbolic structure with totally geodesic boundary on $B^3 - S$, with two rank one cusps corresponding to the tangle strings. We construct T so that its complement admits a hyperbolic structure with totally geodesic boundary, both of whose boundary components match the totally geodesic boundary structure for the complement of S. Links L_n are then obtained by stringing n copies of T along end-to-end, then capping off one end with S and one end with $\bar{S}$. Since all of the boundary identifications are isometries, the geometric structure on the link complement is easily understood in terms of the pieces, and many geometric invariants are easily computed.

One such invariant is the *conformal modulus* of the cusps, a commensurability invaraiant introduced by Thurston in his study of commensurable link complements (Ch 6 of [45]). Chesebro and I compute the conformal modulus of the cusps of the complements of the L_n and show that they are pairwise distinct; hence the complements of the L_n are pairwise incommensurable. Thurston's examples are also distinguished by the conformal moduli of their cusps, but they are distinguished by their trace fields as well.

We would like to extend this construction to find knot complements sharing a trace field. However, it is essential to our construction that we are able to glue together tangles with matching totally geodesic boundaries, as that is what gives control over the algebra. We do not know how to obtain knot complements in this way, in fact we have the following more general question.

Question. *Does there exist a hyperbolic knot complement containing an embedded separating totally geodesic surface?*

To my knowledge, no such surfaces are known, although the subject of totally geodesic surfaces in knot complements has been extensively addressed in the literature. Adams et. al. have given many examples of totally geodesic Seifert surfaces in knot complements [2], [1], Maclachlan identified many examples of immersed totally goedesic surfaces in the Figure 8 knot complement [31], and Aitchison-Rubinstein identified immmersed totally geodesic surfaces in the complements of the "dodecahedral knots" [5]. Much of this work was motivated by the following conjecture of Menasco-Reid.

Conjecture (Menasco-Reid, [34])**.** *No hyperbolic knot complement contains a closed embedded totally geodesic surface.*

Menasco-Reid point out that this conjecture follows from a theorem of Menasco for the class of alternating knots, for topological reasons. Namely, a closed embedded incompressible surface in an alternating knot complement always contains an *accidental parabolic*, that is, an essential curve on the surface which is homotopic into the knot. This is an obstruction to being totally geodesic. There exist similar topological obstructions for many other classes of knots, including Montesinos knots [37], knots of braid index 3 [30] and 4 [33], and 3-bridge knots [24], to name a few. Topological obstructions do not exist in general, however, and examples were given by Adams-Reid of incompressible surfaces in knot complements which are quasifuchsian but not totally geodesic [3]. Recent work of D. Calegari [7] introduces *geometric* obstructions to the existence of totally geodesic surfaces in certain knot complements; he uses them to show that the complement of the knot 8_{20} contains no totally geodesic surfaces at all.

On the other hand, C. Leininger has given evidence for a counterexample to the Menasco-Reid conjecture by constructing a sequence of hyperbolic knot complements containing closed embedded surfaces whose principal curvatures approach 0 [29]. I have given evidence in support of a counterexample from another direction, proving

Theorem 3 (DeBlois, [13])**.** *There exist infinitely many knot complements M_n in hyperbolic rational homology spheres containing closed embedded totally geodesic surfaces.*

The manifolds M_n of the theorem are knot complements in n-fold cyclic covers of a rational homology sphere S branched over a link L. I give a polyhedral decomposition of orbifolds $\mathcal{O}_n$ covered by M_n, obtained from S by orbifold surgery on two components of L. From this decomposition one can find an embedded totally geodesic sphere with 4 cone points in $\mathcal{O}_n$, whose preimage is a closed embedded totally geodesic surface in M_n.

That M_n are rational homology spheres follows from an adaptation of an argument of Sakuma [43] to prove a theorem of Hosokawa-Kinoshita relating the first betti number of cyclic branched covers of links in S^3 to roots of their Alexander polynomials [23]. I believe that Sakuma's argument may be used to directly generalize the theorem of Hosokawa-Kinoshita to the case of link complements in rational homology spheres. I intend to investigate this further.

Nonarithmetic manifolds and character varieties

In my work I have appealed several times to the idea of changing the nature of a manifold by *mutation*, cutting it apart along a totally geodesic hypersurface and regluing by some nontrivial isometry. As a warm-up to proving Theorem 3, I demonstrated the existence of closed hyperbolic rational homology spheres containing an embedded totally geodesic surface by showing that one could destroy homology by mutating a closed hyperbolic manifold along some totally geodesic surface. More recently, Chesebro and I showed that the links of Theorem 2 have integral traces but that mutation along one of the totally geodesic 4-punctured spheres yields link complements possessing nonintegral traces.

A celebrated theorem of Gromov-Piatetski–Shapiro uses a construction related to mutation to show the existence of nonarithmetic real hyperbolic manifolds in all dimensions [19]. Their construction, which they term "interbreeding", produces a nonarithmetic manifold by first constructing arithmetic manifolds M_a and M_b with incompatible arithmetic structures but which possess isometric separating totally geodesic hypersurfaces. Cutting along these surfaces and gluing a remaining component of M_a to a component of M_b yields a nonarithmetic manifold. I wonder if "interbreeding" may be replaced by "inbreeding"—instead of gluing halves of two different arithmetic manifolds together, cut a single arithmetic manifold apart along a totally geodesic subsurface and reglue by some isometry to produce a nonarithmetic manifold. Such a construction would provide a finer version of Gromov-Piatetski–Shapiro, as the manifolds produced by inbreeding would share invariants such as the invariant trace field and quaternion algebra with their arithmetic cousins.

The philosophy behind inbreeding is that arithmetic groups have dense commensurator, so that at least up to passing to covers the boundary has many isometries. Utilizing this philosophy, Ian Agol has shown me a construction that produces nonarithmetic manifolds inbred from subgroups of right-angled Coxeter groups, such as the group of the right-angled 120–cell in $\mathbb{H}^4$. This construction relies on separability properties of these groups to construct covers with a separating totally geodesic suborbifold, a geodesic that meets it at right angles, and an isometry that moves the endpoint of the geodesic a nontrivial but arbitrarily small amount. Cutting apart and regluing by this isometry produces manifolds which must generically be nonarithmetic. I intend to study whether similar constructions may be used with arbitrary arithmetic manifolds to produce nonarithmetic cousins by inbreeding.

For a fixed group G, the *G-character variety* of a group Γ is the set of representations $\Gamma \to G$, up to conjugacy and appropriately topologized. The study of character varieties has proven useful in many contexts for recovering geometric and topological information about hyperbolic manifolds of dimension 2 and 3. Goldman described the topological components of the $\mathrm{PSL}_2(\mathbb{R})$- and $\mathrm{PSL}_2(\mathbb{C})$-character varieties of a surface group—the fundamental group of a closed orientable surface of genus $g \geq 2$—as corresponding to certain Euler and Stiefel-Whitney classes, respectively [18]. He asked whether there are faithful representations in each component. Richard Kent and I have answered this question in the affirmative.

Theorem 4 (DeBlois-Kent, [14])**.** *The set of faithful representations is dense in the* $\mathrm{PSL}_2(\mathbb{R})$*- and* $\mathrm{PSL}_2(\mathbb{C})$*-character varieties of a surface group.*

It is plausible that this theorem could be generalized in several ways, first by extending it to the G-character variety of a surface group for other Lie groups G. For example, Kent and I prove that there exist faithful representations in certain components of the $\mathrm{PU}(2,1)$-character variety of a surface group. I plan to further study which components of the $\mathrm{PU}(n,1)$ character variety contain faithful representations.

One may also expand the class of groups Γ under consideration. A natural next step is the class of *limit groups*, the closure of the set of free groups in the space of marked groups [9], which includes surface groups and free abelian groups. I intend to investigate whether work of Kharlampovich-Myasnikov [25], [26] and Sela [44] (see also Theorem 4.6 of [9]) on the structure of limit groups may be used to give a description of the topological components of their character varieties analogous to Goldman's characterization for surface groups. It may then be possible to use the techniques of Kent and myself to show the existence of faithful representations.

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