

ALGEBRAIC INVARIANTS, MUTATION, AND COMMENSURABILITY OF LINK COMPLEMENTS

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ABSTRACT. We construct a family of hyperbolic link complements, all with trace field $\mathbb{Q}(i, \sqrt{2})$, by gluing tangles along totally geodesic four-punctured spheres, and investigate the commensurability relation among its members. Those with different volume are incommensurable, distinguished by their scissors congruence classes. Mutation produces arbitrarily large finite subfamilies of nonisometric manifolds with the same volume and scissors commensurability class. Depending on the choice of mutation, these manifolds may be commensurable or incommensurable, distinguished in the latter case by cusp parameters. Examples with integral and nonintegral traces are also produced.

1. INTRODUCTION

Manifolds M and M' are *commensurable* if there is a manifold N which is a finite cover for both M and M' . The study of the commensurability relation among hyperbolic knot and link complements in S^3 was initiated by W. Thurston, who described examples of commensurable and incommensurable manifolds in Chapter 6 of his notes [29]. The commensurability relation has been further studied on the chain link complements [22], on two-bridge knot complements [27], and on certain pretzel knot complements [15].

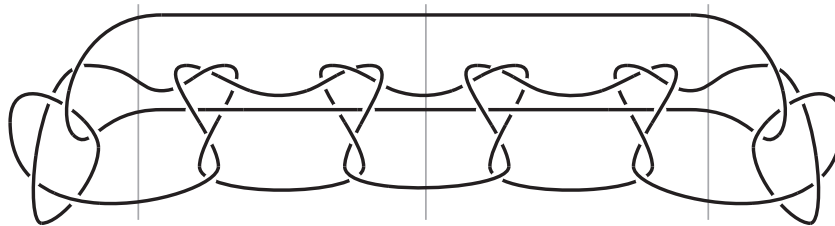
The following algebraic invariants distinguish commensurability classes of link complements under many circumstances. The *trace field* of a hyperbolic manifold $M = \mathbb{H}^3/\Gamma$, $\Gamma < \mathrm{PSL}_2(\mathbb{C})$, is the smallest field containing the traces of elements of Γ . The *cuspidal parameters* of M , used in [29] and [22], are algebraic invariants of the Euclidean structures on horospherical cross-sections of the cusps of a hyperbolic manifold. *Scissors congruence* (cf. [8], [23]) and its relative the *Bloch invariant* [24] are determined by a polyhedral decomposition of a link complement.

In this paper, we construct a family of hyperbolic link complements, upon which the invariants above may be easily computed, and explore the commensurability relation among members of this family.

Theorem 1. *For each $n \in \mathbb{N}$, there is a link $L_n \subset S^3$ such that the complement $M_n = S^3 - L_n$ is hyperbolic with trace field $\mathbb{Q}(i, \sqrt{2})$ and integral traces. If $m \neq n$ then M_m and M_n have distinct Bloch invariants and cusp parameters.*

The link L_2 is pictured in Figure 1. The grey lines in the figure are meant to indicate the presence of 2-spheres $S^{(0)}$, $S^{(1)}$ and $S^{(2)}$ in S^3 , each meeting L_2 in 4 points, which separate it left-to-right into a tangle S in the three-ball B^3 , two copies of a tangle $T \subset S^2 \times I$, and the mirror image \bar{S} of S . For arbitrary $n \in \mathbb{N}$, the link L_n is constructed analogously, using S , \bar{S} , and n copies of T .

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FIGURE 1. The link L_2

It is of fundamental importance here that the tangle complements $M_S = B^3 - S$ and $M_T = S^2 \times I - T$ each admit hyperbolic structures with totally geodesic boundary such that ∂M_S is isometric to a component of ∂M_T . In forming M_n , we have glued $\partial B^3 - S$ to $S^2 \times \{0\} - T$ by a map isotopic to an isometry, so that the separating four-punctured spheres $F^{(i)} = S^{(i)} - T$, $0 \leq i \leq n$, are totally geodesic in M_n , and each copy of M_S and M_T inherits its structure with totally geodesic boundary from the ambient hyperbolic structure. This is one of few examples in which the existence of totally geodesic surfaces in hyperbolic link complements has been proved to occur for geometric, as opposed to topological, reasons. For others, see [16] and [4], and for related results see [19], [14], [1], [2].

The hyperbolic structure with totally geodesic boundary on M_S , which was previously known, is obtained as an identification space of the regular ideal octahedron in \mathbb{H}^3 . The reader will probably have noted that T is itself the double of $T_0 = T \cap (S^2 \times [0, 1/2])$, across $S^2 \times \{1/2\}$. This facilitates gluing by ensuring that the totally geodesic boundary components of M_T are orientation-reversing isometric. We describe M_S and an identification space of the right-angled ideal cuboctahedron producing M_{T_0} in Section 2. In Section 3 we describe assembly of the tangle complements producing the link complements M_n , and in Section 4 we compute the algebraic commensurability invariants described above.

The construction above is a variant of those used by Adams [3] and Neumann-Reid (cf. [17, §5.6]), which produce families of hyperbolic 3-manifolds — in some cases, link complements — by gluing together manifolds bounded by 3-punctured spheres. Unlike the 4-punctured sphere, the 3-punctured sphere is *rigid*: up to isometry, it has a unique complete, finite-volume hyperbolic structure. This automatically implies the analogous total geodesicity property to that described above. A sample result which can be obtained from this construction, due to Neumann-Reid, is that for each imaginary biquadratic extension k of \mathbb{Q} , there are infinitely many commensurability classes of hyperbolic 3-manifolds with trace field k .

Our second concern is the commensurability relation among manifolds related to the M_n by *mutation* along the $F^{(i)}$: cutting along the surface, then regluing by a nontrivial homeomorphism. The mapping class group of a sphere with four marked points has a “mutation subgroup” isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, each member of which is determined by its action as an even permutation of the marked points. We will thus refer to a mutation homeomorphism of a sphere with four marked points by its permutation representation. With L_n projected as in Figure 1, and numbering the spheres separating the copies of S and T left-to-right by $S^{(i)}$, $0 \leq i \leq n$, for each i we mark the points of $S^{(i)} \cap L_n$ by 2, 3, 4, and 1, reading top to bottom.

Since homeomorphisms of S^2 extend over B^3 , if $L \subset S^3$ is a link meeting a two-sphere S in four points, cutting (S^3, L) along $(S, S \cap L)$ and regluing by a mutation yields a link $L' \subset S^3$. We say L' is obtained from L by mutation along S .

In any fixed hyperbolic structure on the 4-punctured sphere, each mutation determines a homeomorphism which is properly isotopic to an isometry [28]. The hyperbolic structure on the $F^{(i)}$, which we describe in Lemma 2.4, corresponds to an arithmetic group $\Lambda < \mathrm{PSL}_2(\mathbb{Z})$ with the property that isometries realizing (13)(24) and (12)(34) lie in different maximal discrete groups containing Λ . We suspect this plays a role in the contrasting behaviors recorded below.

Theorem 2. *For fixed $n \in \mathbb{N}$, each manifold obtained from M_n by mutation along a subcollection of $\{F^{(i)}\}$, using isometries realizing (13)(24), is commensurable to M_n . This produces arbitrarily large finite collections of pairwise nonisometric manifolds as $n \rightarrow \infty$.*

Theorem 2 follows from the fact that the isometry of ∂M_S which realizes (13)(24) extends over M_S but not over M_T ; however the isometry of ∂M_T which realizes (13)(24) on each component lifts to an isometry that extends over a cover of M_T . We describe the isometries realizing (13)(24) and (12)(34) and prove Theorem 2 in Section 5.

That M_n is commensurable with its mutants by (13)(24) implies that it has a large collection of *hidden symmetries*: isometries of a cover which do not lift isometries of M_n . It is known that nonarithmetic manifolds have only finitely many hidden symmetries; however the nature of the collections which may occur remains mysterious.

Mutants of M_n by (12)(34) behave quite differently from mutants by (13)(24), as we will show in Theorem 3. Below, for $n \in \mathbb{N}$ and an element $I \in \{0, 1\}^{n+1}$, we let L_I be the link obtained from L_n by the mutation (12)(34) along $S^{(i)}$, for each i such that the i th entry of I is nonzero. We let $\mathcal{L}_n = \{L_I \mid I \in \{0, 1\}^{n+1}\}$ and write $M_I = S^3 - L_I$ for each I .

Theorem 3. *For every $n \in \mathbb{N}$ we have the following.*

- (1) *For each $I \in \{0, 1\}^{n+1} - \{(0, \dots, 0)\}$, M_I has the same volume, Bloch invariant, and trace field as M_n , but has a nonintegral trace.*
- (2) *There is a subfamily of \mathcal{L}_n with at least $n/2$ mutually incommensurable members, distinguished by their cusp parameters.*
- (3) *There is a subfamily of \mathcal{L}_n with n members which all share cusp parameters.*

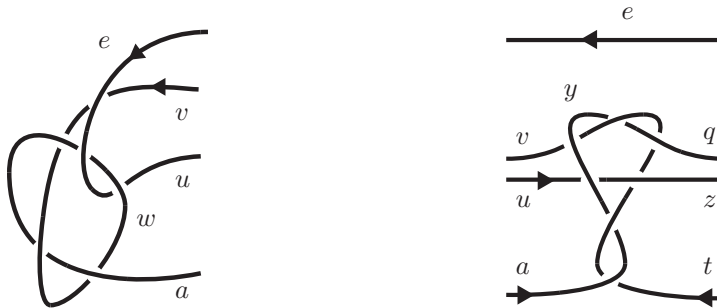
Remarks. 1. Mutation along 4-punctured spheres always preserves hyperbolic volume, by [28], and the trace field, by [21]. It almost certainly preserves the Bloch invariant as well, but we do not know a reference for this assertion.

2. $L_n = L_{(0, \dots, 0)}$, which accounts for the gap in statement (1) of the theorem.

3. We do not know the commensurability relation among members of the subfamily described in (3) above.

We will compute algebraic invariants of the M_I in Section 6, yielding Theorem 3.

Our work is motivated by questions regarding knot complements. Reid-Walsh conjectured that the commensurability class of a hyperbolic knot complement in S^3 contains at most two others [27]. This would imply in particular that any hyperbolic knot complement is incommensurable with most of its mutants. There seems to be no general method for deciding this issue, as mutants are notoriously

FIGURE 2. Tangles S and T_0 , labeled with Wirtinger generators

difficult to distinguish. (Theorem 2 suggests that this may be for good reason!) Methods of Goodman-Heard-Hodgson [9] may be used to decide specific instances; for example, they show that the “Kinoshita-Terasaka knot,” 11n42 in the knot tables, is incommensurable with its mutant, the “Conway knot” 11n34.

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2. A PAIR OF TANGLES

This section is devoted to describing hyperbolic structures with totally geodesic boundary on the complements of the tangles S , in B^3 , and T_0 , in $S^2 \times I$, depicted in Figure 2. For a manifold M with boundary, we refer by a *tangle* in M to a pair (M, T) , where T is the image of a disjoint union of circles and closed intervals, embedded in M by a map taking each circle into the interior of M and restricting on each interval to a proper embedding.

We will prove there is a homeomorphism taking $B^3 - S$ to a hyperbolic manifold with totally geodesic boundary which is an identification space of an ideal octahedron by pairing certain faces. This was previously known, and it follows from results in [25] upon taking a geometric limit, but we do not know a reference for a direct proof. We also prove there is a homeomorphism taking $S^2 \times I - T_0$ to a certain identification space of the right-angled ideal cuboctahedron. As far as we are aware, this description was not previously known.

We prove existence of these homeomorphisms by describing faithful representations from the fundamental groups of the tangle complements to Kleinian groups generated by the face pairings. Our main tools drawing connections between the geometric, algebraic, and topological objects involved are Lemma 2.1, which relates a hyperbolic 3-manifold with totally geodesic boundary produced by pairing some faces of a right-angled polyhedron to the Kleinian group generated by the face pairing isometries, and Lemma 2.7, which describes a homeomorphism between a pared

manifold and the convex core of a Kleinian group to which its fundamental group represents.

In the remainder of the paper, we will let $\mathbb{H}^3 = \{(z, t) \mid z \in \mathbb{C}, t \in (0, \infty)\}$, the upper half space model of hyperbolic space, equipped with the complete Riemannian metric of constant sectional curvature -1 . In this model, the group of orientation-preserving isometries, $\mathrm{PSL}_2(\mathbb{C})$, acts by extending its action by Möbius transformations on the *ideal boundary* or *sphere at infinity* $\mathbb{C} \cup \{\infty\}$.

The *horosphere of height t centered at ∞* is $\mathbb{C} \times \{t\} \subset \mathbb{H}^3$. This inherits the Euclidean metric, scaled by $1/t$, from the ambient hyperbolic metric. For $v \in \mathbb{C} \times \{0\}$, a *horosphere centered at v* is a Euclidean sphere in $\mathbb{C} \times \mathbb{R}$ centered at a point in \mathbb{H}^3 and tangent to $\mathbb{C} \times \{0\}$ at $(v, 0)$. It is a standard fact that isometries of \mathbb{H}^3 take horospheres to horospheres.

A *hyperplane* of \mathbb{H}^3 is a totally geodesic subspace of the form $\ell \times \mathbb{R}^+$ for a line $\ell \subset \mathbb{C}$, or the intersection with \mathbb{H}^3 of a Euclidean sphere centered at a point in $\mathbb{C} \times \{0\}$. A *half space* is the closure of a component of the complement in \mathbb{H}^3 of a hyperplane, and a *polyhedron* is the nonempty intersection of half-spaces. A *face* of a polyhedron is its intersection with one of its defining hyperplanes. A polyhedron is *right-angled* if the defining hyperplanes meet at right angles whenever they meet at all. A polyhedron is *ideal* if any point at which more than two of its defining hyperplanes meet is on the sphere at infinity. Such points are *ideal vertices*.

If $\mathcal{P} \subset \mathbb{H}^3$ is a right-angled ideal polyhedron, we will say that a *checkerboarding* of \mathcal{P} is a partition of the set of faces of \mathcal{P} into sets \mathcal{S}_i and \mathcal{S}_e of “internal” and “external” faces, respectively, so that each $f \in \mathcal{S}_i$ intersects only faces in \mathcal{S}_e and vice-versa. For $f \in \mathcal{S}_i$, let \mathcal{H}_f be the geodesic hyperplane in \mathbb{H}^3 containing f , and let U_f be the half-space determined by \mathcal{H}_f which contains \mathcal{P} . We define the *expansion* of \mathcal{P} as

$$E(\mathcal{P}) = \bigcap_{f \in \mathcal{S}_i} U_f.$$

Then $E(\mathcal{P})$ is a polyhedron of infinite volume containing \mathcal{P} , such that the components of the frontier of \mathcal{P} in $E(\mathcal{P})$ are the external faces of \mathcal{P} .

An *internal face pairing* for a checkerboarded polyhedron \mathcal{P} is a collection $\{\phi_f \mid f \in \mathcal{S}_i\}$ of isometries with the properties that for each $f \in \mathcal{S}_i$, $\phi_f(f) = f'$, where $f' \neq f$ is an internal face of \mathcal{P} , $\phi_f(\mathcal{P}) \cap \mathcal{P} = f'$, and $\phi_{f'} = \phi_f^{-1}$. Given an internal face pairing, if $g \in \mathcal{S}_e$ intersects an internal face f along an edge e , then $f' = \phi_f(f)$ intersects an external face g' along $\phi_f(e)$. Let $M_{\mathcal{P}} \doteq \mathcal{P}/\{\phi_f\}$ be the identification space determined by the equivalence relation $x \sim \phi_f(x)$. Since \mathcal{P} is right-angled, $M_{\mathcal{P}}$ has the structure of a hyperbolic manifold with totally geodesic boundary, where the boundary is the quotient of the disjoint union of the external faces by the edge pairing induced by $\{\phi_f\}$.

If Γ is a Kleinian group — that is, a discrete group of isometries of \mathbb{H}^3 — we will refer by $C(\Gamma)$ to the convex core of \mathbb{H}^3/Γ . This the convex submanifold of \mathbb{H}^3/Γ , minimal with respect to inclusion, with the property that the inclusion-induced homomorphism $\pi_1 C(\Gamma) \rightarrow \mathbb{H}^3/\Gamma$ is surjective. (See the beginning of [20, §6] for background on convex cores.)

Lemma 2.1. *Let $\mathcal{P} \subset \mathbb{H}^3$ be a finite-sided, checkerboarded right-angled ideal polyhedron, with an internal face pairing $\{\phi_f \mid f \in \mathcal{S}_i\}$. Then $\Gamma \doteq \langle \phi_f \mid f \in \mathcal{S}_i \rangle$ is a free Kleinian group, and the inclusion $\mathcal{P} \hookrightarrow \mathbb{H}^3$ induces an isometry $p: M_{\mathcal{P}} \rightarrow C(\Gamma)$.*

Proof. Let $E(\mathcal{P})$ be the expansion of \mathcal{P} . The internal face pairing for \mathcal{P} determines a genuine face pairing for $E(\mathcal{P})$, such that the inclusion $\mathcal{P} \rightarrow E(\mathcal{P})$ induces an isometric embedding $M_{\mathcal{P}} \rightarrow E(\mathcal{P})/\{\phi_f\}$. If $E(\mathcal{P})/\{\phi_f\}$ is a complete hyperbolic 3-manifold, then by Poincaré's polyhedron theorem (see eg. [26, Theorem 11.2.2]), $\Gamma = \langle \phi_f \mid f \in \mathcal{S}_i \rangle$ is discrete and $E(\mathcal{P})$ is a fundamental domain for Γ .

We will use the terminology and some results of [26]. By [26, Theorem 11.1.6], to show completeness it suffices to check that the link of any cusp is a complete Euclidean surface. Let $[v] = \{v_0, v_1, \dots, v_{n-1}\}$ be an equivalence class of ideal vertices of \mathcal{P} under the relation generated by $x \sim \phi_f(x)$, $f \in \mathcal{S}_i$, enumerated so that for each j there exists $f_j \in \mathcal{S}_i$ with $\phi_{f_j}(v_j) = v_{j+1}$ (taken modulo n). In particular, v_j is an ideal vertex of f_j and also of $f'_j \doteq \phi_{j-1}(f_{j-1})$.

For each j , let \mathcal{B}_j be a horosphere centered at v_j , chosen small enough that $\mathcal{B}_j \cap \mathcal{B}_{j'} = \emptyset$ for $j \neq j'$. Since \mathcal{P} is right-angled, $\mathcal{B}_j \cap \mathcal{P}$ is a Euclidean rectangle for each j . We may assume, by renumbering if necessary, that $\mathcal{B}_0 \cap f_0$ has shortest length of all the arcs $\mathcal{B}_j \cap f_j$. Then since $\phi_0(\mathcal{B}_0) \cap f'_1$ is parallel to $\phi_0(\mathcal{B}_0) \cap f_1$ in $\phi_0(\mathcal{B}_0) \cap \mathcal{P}$, they have the same length: that of $\mathcal{B}_0 \cap f_0$. Since this is less than the length of $\mathcal{B}_1 \cap f_1$, we have $\phi_0(\mathcal{B}_0) \subset \mathcal{B}_1$.

We may replace \mathcal{B}_1 by $\phi_0(\mathcal{B}_0)$, then replace \mathcal{B}_2 with $\phi_1(\mathcal{B}_1)$ and so on, yielding a new collection of horospheres which are pairwise disjoint and have the additional property that they are interchanged by the face pairings of \mathcal{P} . Equivalence classes of ideal vertices of $E(\mathcal{P})$ are the same as those of \mathcal{P} ; thus this collection satisfies the hypotheses of [26, Theorem 11.1.4], and the link of $[v]$ is complete. It follows that $E(\mathcal{P})/\{\phi_f\}$ is a complete hyperbolic 3-manifold.

Now by the polyhedron theorem, Γ is discrete and $E(\mathcal{P})$ is a fundamental domain for Γ . It follows from a ping-pong argument that Γ is free, since the fact that \mathcal{P} is right-angled implies that the hyperplanes containing its internal faces are mutually disjoint. The inclusion $E(\mathcal{P}) \rightarrow \mathbb{H}^3$ induces an isometry $E(\mathcal{P})/\{\phi_f\} \rightarrow \mathbb{H}^3/\Gamma$, so the inclusion $\mathcal{P} \rightarrow \mathbb{H}^3$ induces an isometric embedding $M_{\mathcal{P}} \rightarrow \mathbb{H}^3/\Gamma$.

For an external face g of \mathcal{P} , let \mathcal{H}_g be the hyperplane of \mathbb{H}^3 containing g , and let V_g be the closure of the component of $\mathcal{Q} \cap (\mathbb{H}^3 - \mathcal{H}_g)$ which does not intersect \mathcal{P} . There is a homeomorphism $V_g \rightarrow g \times [0, \infty)$ taking x to $(r_g(x), d(x, r_g(x)))$, where $r_g: \mathbb{H}^3 \rightarrow \mathcal{H}_g$ is the nearest-point retraction and $d(x, y)$ is the distance in \mathbb{H}^3 between x and y .

If g intersects an internal face f along an edge e , then ϕ_f takes \mathcal{H}_g to $\mathcal{H}_{g'}$, where g' intersects $f' = \phi_f(f)$ along $e' = \phi_f(e)$. Furthermore, $r_g^{-1}(e) \subset V_g$ is taken to $r_{g'}^{-1}(e')$, equivariantly with respect to the homeomorphisms to $e \times [0, \infty)$ and $e' \times [0, \infty)$. It follows that \mathbb{H}^3/Γ is homeomorphic to $M_{\mathcal{P}} \cup_{\partial} \partial M_{\mathcal{P}} \times [0, \infty)$, so $M_{\mathcal{P}}$ contains $C(\Gamma)$.

It now follows from work of Kojima ([12], see also [13]) that the convex hull of Γ_S has boundary consisting of translates of the hyperplanes containing external faces of \mathcal{P} . These cover $\partial M_{\mathcal{P}}$, and so $p(M_{\mathcal{P}}) = C(\Gamma_S)$. \square

Corollary 2.2. *Let \mathcal{P}_1 be the regular ideal octahedron in \mathbb{H}^3 , embedded as indicated in Figure 2, and checkerboarded by declaring the indicated face A to be external. The collection $\{s^{\pm 1}, t^{\pm 1}\}$ is an internal face pairing for \mathcal{P}_1 , where*

$$s = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 2i & 2-i \\ i & 1-i \end{pmatrix}.$$

Let $M_S = \mathcal{P}/\{\mathbf{s}^{\pm 1}, \mathbf{t}^{\pm 1}\}$, and let $\Gamma_S = \langle \mathbf{s}, \mathbf{t} \rangle$. Then the inclusion $\mathcal{P} \rightarrow \mathbb{H}^3$ induces an isometry $p_S: M_S \rightarrow C(\Gamma_S)$.

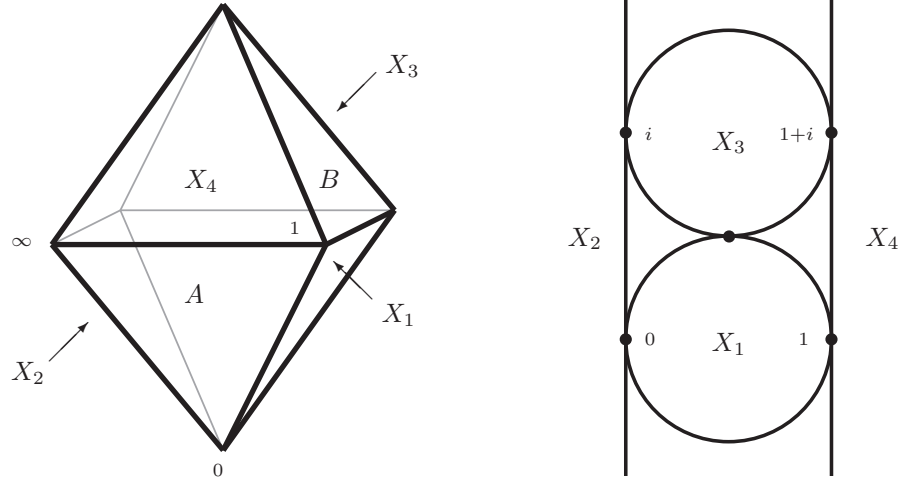


FIGURE 3. The regular ideal octahedron \mathcal{P}_1 , and its expansion $E(\mathcal{P}_1)$

Proof. With the indicated embedding, \mathcal{P}_1 is a tile of the $\mathrm{PSL}_2(\mathcal{O}_1)$ -invariant tessellation \mathcal{T}_1 constructed in [10]. Here $\mathcal{O}_1 = \mathbb{Z}[i]$ is the ring of integers of the field $\mathbb{Q}(i)$. In particular, the face A shown on the left in Figure 2 to have ideal vertices 0 , 1 , and ∞ , and all other ideal vertices of \mathcal{P}_1 have positive imaginary part.

Since A is external, the faces X_1 , X_2 , X_3 , and X_4 of \mathcal{P}_1 indicated on the left in Figure 2 are internal. Direct computation reveals that \mathbf{s} takes X_1 to X_2 , fixing the ideal vertex they share, and \mathbf{t} takes X_3 to X_4 so that the vertex they share goes to the vertex shared by X_4 and X_2 . Hence $\{\mathbf{s}^{\pm 1}, \mathbf{t}^{\pm 1}\}$ is an internal face pairing for \mathcal{P}_1 . The corollary now follows from Lemma 2.1. \square

The external faces of \mathcal{P}_1 triangulate ∂M_S , and their images under p_S determine a triangulation of $\partial C(\Gamma_S)$, which we will denote by Δ_S .

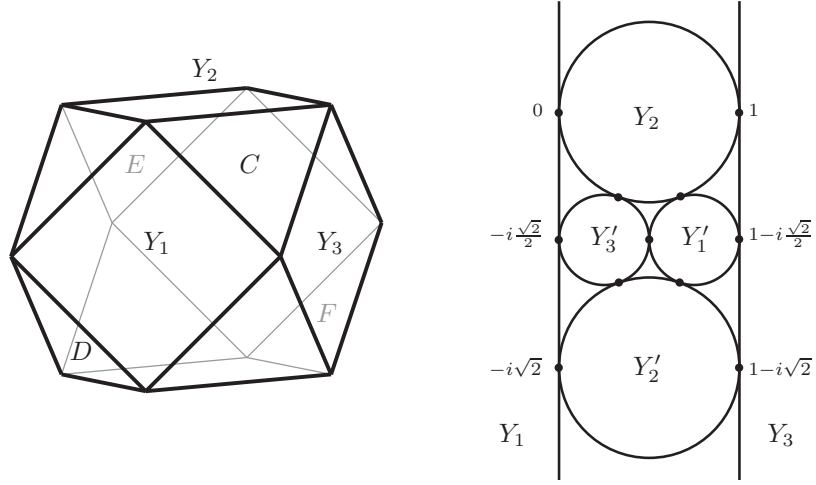
Corollary 2.3. *Let \mathcal{P}_2 be the right-angled ideal cuboctahedron in \mathbb{H}^3 , embedded as indicated in Figure 4, and checkerboarded by declaring triangular faces external. The collection $\{\mathbf{f}^{\pm 1}, \mathbf{g}^{\pm 1}, \mathbf{h}^{\pm 1}\}$ is an internal face pairing for \mathcal{P}_2 , where*

$$\mathbf{f} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \mathbf{g} = \begin{pmatrix} -1+i\sqrt{2} & 1-2i\sqrt{2} \\ -2 & 3-i\sqrt{2} \end{pmatrix} \quad \mathbf{h} = \begin{pmatrix} 2i\sqrt{2} & -3-i\sqrt{2} \\ -3+i\sqrt{2} & -3i\sqrt{2} \end{pmatrix}.$$

Let $M_{T_0} = \mathcal{P}_2/\{\mathbf{f}^{\pm 1}, \mathbf{g}^{\pm 1}, \mathbf{h}^{\pm 1}\}$, and let $\Gamma_{T_0} = \langle \mathbf{f}, \mathbf{g}, \mathbf{h} \rangle$. The inclusion $\mathcal{P}_2 \rightarrow \mathbb{H}^3$ induces an isometry $p_{T_0}: M_{T_0} \rightarrow C(\Gamma_{T_0})$.

Proof. With the indicated embedding, \mathcal{P}_2 is a tile of the $\mathrm{PSL}_2(\mathcal{O}_2)$ -invariant tessellation \mathcal{T}_2 of \mathbb{H}^3 defined in [10], where $\mathcal{O}_2 = \mathbb{Z}[i\sqrt{2}]$ is the ring of integers of $\mathbb{Q}(i\sqrt{2})$. In particular, the face C labeled in the figure has ideal vertices 0 , 1 , and ∞ .

Label the internal faces Y_i as indicated on the left in Figure 4, and label the square face opposite Y_i as Y'_i . Direct computation reveals that \mathbf{f} takes Y_2 to Y_1 , fixing the ideal vertex they share, \mathbf{g} takes Y_3 to Y'_1 , fixing the ideal vertex they

FIGURE 4. The right-angled ideal cuboctahedron \mathcal{P}_2 , and $E(\mathcal{P}_2)$

share, and h takes Y'_2 to Y'_3 , taking the vertex they share to the opposite vertex on Y'_3 . Hence $\{f^{\pm 1}, g^{\pm 1}, h^{\pm 1}\}$ is an internal face pairing for \mathcal{P}_2 . The corollary now follows from Lemma 2.1. \square

The external faces of \mathcal{P}_2 triangulate ∂M_{T_0} . We will call $\partial_- M_{T_0}$ the component which is triangulated by the faces labeled in Figure 4 with letters, and $\partial_+ M_{T_0}$ the other component. Let $\partial_{\pm} C(\Gamma_{T_0}) = p_{T_0}(\partial_{\pm} M_{T_0})$ and let $\Delta_{T_0}^{\pm}$ refer to the triangulation for $\partial_{\pm} C(\Gamma_{T_0})$ determined by the images of the external faces of \mathcal{P} under p_{T_0} .

In the remainder of the paper, if g and h are elements of a group and G is a subgroup, we let g^h denote the conjugate of g by h , hgh^{-1} and $G^h = hGh^{-1}$.) Below we describe parabolic isometries p_1 , p_2 and p_3 which lie in $\Gamma_S \cap \Gamma_{T_0}$.

$$\begin{aligned} p_1 &= s^{-1} = f^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ p_2 &= stst^{-2} = fg^{-1}f^{-1}h^{-1}g = \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix} \\ p_3 &= (s^{-1})^tst = (g^{-1})g^{-1}f^{-1}h = \begin{pmatrix} -14 & 25 \\ -9 & 16 \end{pmatrix}. \end{aligned}$$

Since p_1 , p_2 , and p_3 are in $\mathrm{PSL}_2(\mathbb{R})$, they stabilize the hyperplane \mathcal{H} with ideal boundary $\mathbb{R} \cup \{\infty\}$.

Lemma 2.4. *Let $\Lambda = \langle p_1, p_2, p_3 \rangle < \mathrm{PSL}_2(\mathbb{R})$. Then the polyhedron \mathcal{F} of Figure 5 is a fundamental domain for Λ , and $F^{(0)} = \mathcal{H}/\Lambda$ is a 4-punctured sphere. Furthermore:*

- (1) $\Lambda = \mathrm{Stab}_{\Gamma_S}(\mathcal{H}) = \mathrm{Stab}_{\Gamma_{T_0}}(\mathcal{H})$,
- (2) the inclusion $\mathcal{H} \hookrightarrow \mathbb{H}^3$ induces an isometry $\iota_-^{(0)}: F^{(0)} \rightarrow \partial C(\Gamma_S)$ and an isometry $\iota_+^{(0)}: F^{(0)} \rightarrow \partial_- C(\Gamma_{T_0})$, and
- (3) the triangulation of \mathcal{F} pictured in Figure 5 projects to a triangulation Δ_F of $F^{(0)}$ taken by $\iota_-^{(0)}$ and $\iota_+^{(0)}$, respectively, to Δ_S and $\Delta_{T_0}^-$.

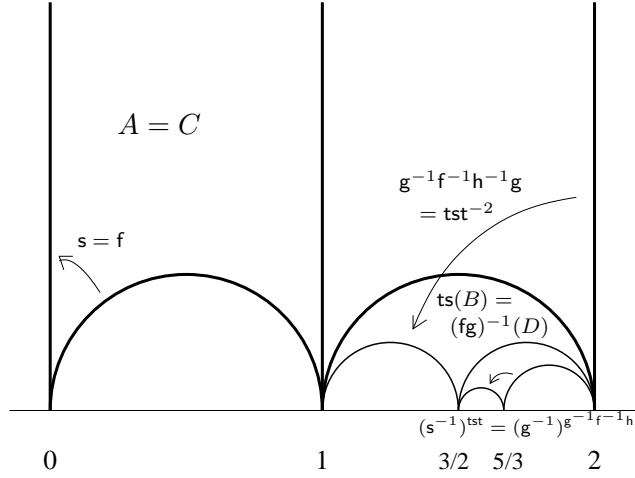


FIGURE 5. A fundamental domain \mathcal{F} for the action of Λ on \mathbf{H} , and side pairings.

Proof. Let A and B be the external faces of \mathcal{P}_1 labeled in Figure 2, let A' be the external face which shares the ideal vertex 0 with A , and let B' be the other external face, which shares the vertex ∞ with A and $1+i$ with B . Let C , D , E , and F be the external faces of \mathcal{P}_2 indicated in Figure 4. With the prescribed embeddings of \mathcal{P}_1 and \mathcal{P}_2 , their faces A and C coincide and lie in \mathcal{H} .

Each triangle of the domain \mathcal{F} pictured in Figure 5 is a Γ_S -translate of a face of \mathcal{P}_1 , and a Γ_{T_0} -translate of a face of \mathcal{P}_2 . For example, since t takes X_3 to X_4 , with the edge $X_3 \cap B'$ taken to $X_4 \cap A$, it follows that $t(B')$ lies in \mathcal{H} , abutting A along the geodesic between 1 and ∞ . Similarly, $g^{-1}(E)$ abuts $C = A$ along same edge.

Direct calculation verifies that $t(B') = g^{-1}(E)$ is the triangle whose other vertex is at 2. This also follows from the fact noted in the proofs of Corollaries 2.2 and 2.3, that Γ_S -translates of \mathcal{P}_1 lie in the tessellation \mathcal{T}_1 described in [10], and Γ_{T_0} -translates of \mathcal{P}_2 lie in \mathcal{T}_2 , and each of these intersects \mathcal{H} in the Farey tessellation. One further finds that $ts(B) = (fg)^{-1}(D)$, as indicated in Figure 5, and $tst(A') = g^{-1}f^{-1}h(F)$ has vertices at $3/2$, 2, and $5/3$.

Combinatorial considerations or direct calculation establish that $s = f$, $tst^{-2} = g^{-1}f^{-1}h^{-1}g$, and $(s^{-1})^{tst} = (g^{-1})^{g^{-1}f^{-1}h^{-1}}$, and that each stabilizes \mathcal{H} and pairs edges of \mathcal{F} as indicated in Figure 5. Inspection reveals that the quotient of \mathcal{F} by these edge pairings is a 4-punctured sphere. By the polyhedron theorem, \mathcal{F} is a fundamental domain for the group that they generate, and its quotient by the edge pairings is the quotient of \mathcal{H} by this group. Since p_1 , p_2 , and p_3 are easily obtained from the edge pairings above and vice-versa, it follows that

$$\Lambda = \langle s, tst^{-2}, (s^{-1})^{tst} \rangle = \langle f, g^{-1}f^{-1}h^{-1}g, (g^{-1})^{g^{-1}f^{-1}h^{-1}} \rangle.$$

Therefore \mathcal{F} is a fundamental domain for Λ , and \mathcal{H}/Λ is a four-punctured sphere $F^{(0)}$.

It is clear from the descriptions of its generators that Λ is contained in $\text{Stab}_{\Gamma_S}(\mathcal{H})$ and $\text{Stab}_{\Gamma_{T_0}}(\mathcal{H})$, so the inclusion $\mathcal{H} \rightarrow \mathbb{H}^3$ induces immersions $\iota_-^{(0)}: F^{(0)} \rightarrow \mathbb{H}^3/\Gamma_S$

and $\iota_+^{(0)} : F^{(0)} \rightarrow \mathbb{H}^3/\Gamma_{T_0}$ which factor through covering maps to $\mathcal{H}/\text{Stab}_{\Gamma_S}(\mathcal{H})$ and $\mathcal{H}/\text{Stab}_{\Gamma_{T_0}}(\mathcal{H})$, respectively.

Since \mathcal{H} contains the external face A of \mathcal{P}_1 , the projection to \mathbb{H}^3/Γ_S sends it to a component of $\partial C(\Gamma_S)$. Since it contains C , the projection to $\mathbb{H}^3/\Gamma_{T_0}$ sends it to a component of $\partial C(\Gamma_{T_0})$. Hence $\iota_-^{(0)}$ and $\iota_+^{(0)}$ are covering maps to components of $\partial C(\Gamma_S)$ and $\partial C(\Gamma_{T_0})$, respectively. But by Lemma 2.1, these components are triangulated by the external faces of \mathcal{P}_1 and \mathcal{P}_2 described above, whence conclusions (1), (2), and (3) follow immediately. \square

Remarks. 1. The parabolic elements of Λ which fix the ideal vertices $0, \infty$, and $5/3$ of \mathcal{F} are $\mathfrak{p}_1, \mathfrak{p}_2$, and \mathfrak{p}_3 . The final conjugacy class of parabolic elements in Λ is represented by

$$\mathfrak{p}_4 = \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3^{-1} = (\text{stst}^{-2})^{\text{tst}^{-1}} = \begin{pmatrix} 29 & -45 \\ 20 & -31 \end{pmatrix}.$$

Evidently \mathfrak{p}_1 and \mathfrak{p}_3 are conjugate in Γ_S , as are \mathfrak{p}_2 and \mathfrak{p}_4 . The combinatorial considerations of Lemma 4.10 will show that $C(\Gamma_S)$ has exactly two cusps, each of rank one, so every parabolic element of Γ_S is conjugate to one of \mathfrak{p}_1 or \mathfrak{p}_2 .

2. There exists $\mathfrak{k} \in \text{PSL}_2(\mathbb{C})$, with order 2, which normalizes Γ_{T_0} :

$$(1) \quad \mathfrak{k} = \begin{pmatrix} i & i - \sqrt{2} \\ 0 & -i \end{pmatrix}.$$

The action of \mathfrak{k} on the generators $\mathfrak{f}, \mathfrak{g}$, and \mathfrak{h} is given by

$$\mathfrak{f}^{\mathfrak{k}} = \mathfrak{g}^{\mathfrak{f}\mathfrak{g}^{-1}}, \quad \mathfrak{g}^{\mathfrak{k}} = \mathfrak{f}\mathfrak{g}^{-1}, \quad \text{and} \quad \mathfrak{h}^{\mathfrak{k}} = (\mathfrak{h}^{-1})^{\mathfrak{f}\mathfrak{g}^{-1}}.$$

Definition 2.5. If Γ is a Kleinian group and $\mathfrak{u} \in \text{Isom}(\mathbb{H}^3)$, we write $\phi_{\mathfrak{u}} : C(\Gamma) \rightarrow C(\Gamma^{\mathfrak{u}})$ for the induced isometry. Note that $\phi_{\mathfrak{u}}\phi_{\mathfrak{v}} = \phi_{\mathfrak{uv}} : C(\Gamma) \rightarrow C(\Gamma^{\mathfrak{uv}})$.

Since \mathfrak{k} normalizes Γ_{T_0} , $\phi_{\mathfrak{k}} : C(\Gamma_{T_0}) \rightarrow C(\Gamma_{T_0})$ acts as an orientation-preserving involution. The elements $\mathfrak{p}_i^{\mathfrak{k}}, i \in \{1, 2, 3\}$, all preserve the geodesic hyperplane $\mathfrak{k}(\mathcal{H})$ which lies over the line $\mathbb{R} - i\sqrt{2}$. This hyperplane contains an external face of \mathcal{P}_2 which projects to $\partial_+ C(\Gamma_{T_0})$. The lemma below follows and, together with Lemma 2.4, completely describes $\partial C(\Gamma_{T_0})$.

Lemma 2.6. *Let $F' = \mathfrak{k}(\mathcal{H})/\Lambda^{\mathfrak{k}}$. Then $\Lambda^{(\mathfrak{k})} = \text{Stab}_{\Gamma_T}(\mathfrak{k}(\mathcal{H}))$, and the inclusion $\mathfrak{k}(\mathcal{H}) \rightarrow \mathbb{H}^3$ induces an isometry $F' \rightarrow \partial_+ C(\Gamma_{T_0})$.*

It is easy to see that $\mathfrak{p}_1^{\mathfrak{k}}$ is conjugate in Γ_{T_0} to \mathfrak{p}_3 and vice-versa, and that $\mathfrak{p}_2^{\mathfrak{k}} = \mathfrak{p}_2^{-1}$. The combinatorial considerations of Lemma 4.11 will imply that M_{T_0} has four cusps. Hence by Lemma 2.6, each of the cusps of $C(\Gamma_{T_0})$ joins $\partial_- C(\Gamma_{T_0})$ to $\partial_+ C(\Gamma_{T_0})$, and each parabolic in Γ_{T_0} is conjugate to exactly one $\mathfrak{p}_i, i \in \{1, 2, 3, 4\}$.

Our second main tool in this section is Lemma 2.7 below. We refer to [20, Definition 4.8] for the definition of a pared manifold.

Lemma 2.7. *Let (M, P) be a pared manifold, and suppose that $\rho : \pi_1 M \rightarrow \Gamma < \text{PSL}_2(\mathbb{C})$ is a faithful representation onto a non-Fuchsian geometrically finite Kleinian group where $C(\Gamma)$ has totally geodesic boundary. If there is a one-to-one correspondence under ρ between conjugacy classes of $\pi_1(M)$ corresponding to elements of P and conjugacy classes of maximal parabolic subgroups of Γ , then ρ is induced by a homeomorphism of $M - P$ to $C(\Gamma)$.*

This is well known to experts in Kleinian groups, but we do not know of a reference for a written proof. It seems worth writing down as it may fail if $C(\Gamma)$ does not have totally geodesic boundary (see [7] for a thorough exploration of this phenomenon). The proof follows easily from results in [7] for example, but requires introduction of the characteristic submanifold machinery. Since this is somewhat outside the scope of the rest of the paper, we defer the proof to Appendix A.

Proposition 2.8. *Let (B^3, S) be the tangle pictured on the left in Figure 2. There is a homeomorphism $f_S: B^3 - S \rightarrow C(\Gamma_S)$, such that*

$$f_{S*}: \pi_1(B^3 - S) \rightarrow \Gamma_S$$

is given by $f_{S}(a) = \mathfrak{p}_1^{-1}$, $f_{S*}(e) = \mathfrak{p}_2$, and $f_{S*}(v) = \mathfrak{p}_3^{-1}$. Here we take a base point for $\pi_1(B^3 - S)$ on $\partial(B^3 - S)$ high above the projection plane, and let Wirtinger generators correspond in the usual way to labeled arcs of the diagram.*

Proof. Reducing a standard Wirtinger presentation for $\pi_1(B^3 - S)$, we obtain

$$\left\langle a, w, e \mid ewe^{-1}a = awaw^{-1} \right\rangle = \left\langle a, w, e \mid w(e^{-1}aw) = (e^{-1}aw)a \right\rangle$$

Thus taking $b = e^{-1}aw$, one finds that $\pi_1(B^3 - S)$ is freely generated by a and b .

By Lemma 2.1 and Corollary 2.2, Γ_S is free on the generators \mathfrak{s} and \mathfrak{t} . Hence, the map $f_{S*}: \pi_1(B^3 - S) \rightarrow \Gamma_S$ given by $a \mapsto \mathfrak{s}$ and $b \mapsto \mathfrak{t}$ is an isomorphism. Notice that the subgroup of $\pi_1(B^3 - S)$ corresponding to the 4-punctured sphere $\partial B^3 - \partial S$ is freely generated by a , v , and e . It is easily checked that

$$f_{S*}(v) = \begin{pmatrix} 16 & -25 \\ 9 & -14 \end{pmatrix} = \mathfrak{p}_3^{-1}$$

and

$$f_{S*}(e) = \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix} = \mathfrak{p}_2.$$

f_{S*} takes $\pi_1(\partial B^3 - S)$ isomorphically to Λ , since a , v , and e generate $\pi_1(\partial B^3 - S)$ and their images in Γ_S generate Λ . Since any meridian of S is conjugate in $\pi_1(B^3 - S)$ to either a or e , and these are taken to \mathfrak{p}_1 and \mathfrak{p}_2 respectively, meridians are taken to parabolic elements of Γ_S .

Now let $N(S)$ be a small open tubular neighborhood of S in B^3 . Then $B^3 - N(S)$ is a compact manifold with genus two boundary, and the pair $(B^3 - N(S), \partial N(S))$ is a pared manifold. The proposition follows from Lemma 2.7, after noting that $(B^3 - N(S)) - \partial N(S)$ is homeomorphic to $B^3 - S$. \square

The proposition below is the analog for T_0 of Proposition 2.8.

Proposition 2.9. *Let $(S^2 \times I, T_0)$ be the tangle pictured on the right side of Figure 2, where I is oriented so that $\partial_- T_0 \doteq T_0 \cap S^2 \times \{0\}$ contains the endpoints labeled a , u , and v . There is a homeomorphism $f_{T_0}: S^2 \times I - T_0 \rightarrow C(\Gamma_{T_0})$ such that*

$$f_{T_0*}: \pi_1(S^2 \times I - T_0) \rightarrow \Gamma_{T_0}$$

is given by $f_{T_0}(a) = \mathfrak{p}_1^{-1}$, $f_{T_0*}(e) = \mathfrak{p}_2$, and $f_{T_0*}(v) = \mathfrak{p}_3^{-1}$. Here we take a base point for $\pi_1(S^2 \times I - T_0)$ on $S^2 \times \{0\}$ high above the projection plane and let Wirtinger generators correspond to the labeled arcs of Figure 2.*

Proof. ($S^2 \times I - N(T_0)$) may be isotoped in S^3 to a standard embedding of a genus-3 handlebody. Thus $\pi_1(S^2 \times I - T_0)$ is free on three generators. We claim that the group is generated by a , e , and t . This follows after noticing that $v = y^{-1}xy$ where $y = (ta)^{-1}a(ta)$ and $x = (azq)^{-1}t(azq) = (ate)^{-1}t(ate)$. (The relation $zq = te$ used in the last equality comes from the relationship between four peripheral elements in a 4-punctured sphere group.) So far, we have established that $v, y \in \langle a, e, t \rangle$. Now using the other punctured sphere relation, we have $u = a^{-1}ev \in \langle a, e, t \rangle$. Finally, $z = yuy^{-1}$ and $q = z^{-1}te$. Therefore a , e , and t generate the group as claimed.

By Lemma 2.1 and Corollary 2.3, Γ_{T_0} is freely generated by \mathbf{f} , \mathbf{g} , and \mathbf{h} . For our purposes, a more convenient free generating set for Γ_{T_0} is $\{\mathbf{f}, \mathbf{fgf}^{-1}, \mathbf{p}_2\}$. Note that all of these generators are parabolic and peripheral, and conjugation by \mathbf{k} interchanges the first two and takes the third to its inverse. The representation of $\pi_1(S^2 \times [0, 1/2] - T_0)$ given by

$$a \mapsto \mathbf{f} \qquad t \mapsto \mathbf{fgf}^{-1} \qquad e \mapsto \mathbf{p}_2$$

is clearly faithful, and it is easily checked that v maps to \mathbf{p}_3^{-1} . Because $u = a^{-1}ev$ is mapped to $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3^{-1} = \mathbf{p}_4$, we conclude that meridians are mapped to parabolic elements and that $\pi_1(S^2 \times \{0\} - \partial_-T_0)$ is taken to Λ . The result now follows from Lemma 2.7 as previously. \square

There is a visible involution of $S^2 \times I - T_0$ which is a rotation by π around a circle in $S^2 \times \{1/2\}$. This involution exchanges the two boundary components. With a proper choice of path between our basepoint and its image under this involution, the corresponding action on $\pi_1(S^2 \times I - T_0)$ is given by

$$a \leftrightarrow t \qquad e \leftrightarrow e^{-1}$$

This commutes with the action of the element \mathbf{k} defined in (1) on Γ_{T_0} , under the representation f_{T_0*} . Hence this involution is isotopic to the pullback of $\phi_{\mathbf{k}}$ by f_{T_0} .

Recall from Lemma 2.4 that $\Lambda = \text{Stab}_{\Gamma_{T_0}}(\mathcal{H})$, and from Lemma 2.6 that $\Lambda^{\mathbf{k}} = \text{Stab}_{\Gamma_{T_0}}(\mathbf{k}(\mathcal{H}))$. By its definition in Proposition 2.9, it is clear that f_{T_0*} maps $\pi_1 S^2 \times \{0\} - \partial_-T_0$ isomorphically to Λ . Then since \mathcal{H} projects to $\partial_-C(\Gamma_{T_0})$, using the involution equivariance of f_{T_0} we obtain the corollary below.

Corollary 2.10. *Let $\partial_+T_0 = T_0 \cap S^2 \times \{1\}$. Then $f_{T_0}(S^2 \times \{0\} - \partial_-T_0) = \partial_-C(\Gamma_{T_0})$, and $f_{T_0}(S^2 \times \{1\} - \partial_+T_0) = \partial_+C(\Gamma_{T_0})$.*

3. COMBINATION

In this section, we will describe how to join copies of the tangles S and T_0 to construct links in S^3 whose complements are uniformized by combinations of Γ_S and Γ_{T_0} . The main tool in this section is a corollary of Maskit's combination theorem for free products with amalgamation [18]. Below, let $\text{Hull}(\Gamma)$ denote the convex hull of the limit set for a Kleinian group Γ .

Definition 3.1. Kleinian groups Γ_0 and Γ_1 *meet cute* along a hyperplane $\mathcal{K} \subset \mathbb{H}^3$ if $\mathcal{K} = \text{Hull}(\Gamma_0) \cap \text{Hull}(\Gamma_1)$ and $\text{Stab}_{\Gamma_0}(\mathcal{K}) = \text{Stab}_{\Gamma_1}(\mathcal{K})$.

The utility of this definition is due to the following fact, an easy consequence of the definitions.

Fact. *If Γ_0 and Γ_1 meet cute along \mathcal{K} , \mathcal{K} divides \mathbb{H}^3 into \mathcal{B}_0 and \mathcal{B}_1 , such that:*

- (1) $\Gamma_0 \cap \Gamma_1$ stabilizes \mathcal{K} .

(2) For $i \in \{0, 1\}$, if $\mathfrak{g}_i \in \Gamma_i$ satisfies $\mathfrak{g}_i(\mathcal{B}_{1-i}) \cap \mathcal{B}_{1-i} \neq \emptyset$, then $\mathfrak{g}_i \in \Gamma_0 \cap \Gamma_1$.

In general, if Θ is a subgroup of Γ , the limit set of Θ is contained in that of Γ , and so the covering map $\mathbb{H}^3/\Theta \rightarrow \mathbb{H}^3/\Gamma$ maps $C(\Theta)$ into $C(\Gamma)$ — we will call this restriction the *natural map* $C(\Theta) \rightarrow C(\Gamma)$. When Γ_0 and Γ_1 meet cute along \mathcal{K} then the natural map $C(\Gamma_0 \cap \Gamma_1) \rightarrow C(\Gamma_i)$ restricts to an embedding of the 2-orbifold $\mathcal{K}/(\Gamma_0 \cap \Gamma_1)$.

The lemma below is a geometric combination theorem for Kleinian groups which meet cute along a hyperplane. It follows from Maskit's combination theorem and an observation due to Morgan.

Lemma 3.2. *Suppose Γ_0 and Γ_1 meet cute along a plane \mathcal{K} . Let $E = \mathcal{K}/\Theta$, where $\Theta = \Gamma_0 \cap \Gamma_1$, and for $i = 0, 1$ let $\iota_i: E \rightarrow C(\Gamma_i)$ be the natural embedding. Then $\langle \Gamma_0, \Gamma_1 \rangle$ is a Kleinian group, and the inclusions $\Gamma_i \rightarrow \langle \Gamma_0, \Gamma_1 \rangle$ determine an isomorphism $\Gamma_0 *_{\Theta} \Gamma_1 \rightarrow \langle \Gamma_0, \Gamma_1 \rangle$ as abstract groups. The natural maps $C(\Gamma_i) \rightarrow C(\langle \Gamma_0, \Gamma_1 \rangle)$ determine an isometry $C(\Gamma_0) \cup_{\iota_1 \iota_0^{-1}} C(\Gamma_1) \rightarrow C(\langle \Gamma_0, \Gamma_1 \rangle)$.*

In using Lemma 3.2, we often write $C(\Gamma_0) \cup_E C(\Gamma_1)$ when the maps ι_i are clear.

Proof. We will use the version of Maskit's combination theorem recorded in [20, Theorem 8.2]). The fact above implies that Γ_0 and Γ_1 which meet cute along a hyperplane \mathcal{K} satisfy the hypotheses of [20, Theorem 8.2], whence the group-theoretic conclusions of the lemma follow. The existence of the required isometry follows from the remarks in [20] below Theorem 8.2, noting that in this case the surface $X = f^{-1}(\frac{1}{2})$ is the totally geodesic surface uniformized by Θ . \square

Our first application of Lemma 3.2 describes a Kleinian group with totally geodesic convex core boundary isometric to the double of $C(\Gamma_{T_0})$ across $\partial_+ C(\Gamma_{T_0})$. If M is an oriented manifold with a boundary component F , the *double* of M across F is $M \cup_F \overline{M}$, where \overline{M} is a copy of M with orientation reversed, and the gluing map $F \rightarrow \overline{F} \subset \overline{M}$ is the identity map.

Recall from above Lemma 2.4 that we have defined \mathcal{H} to be the geodesic hyperplane of \mathbb{H}^3 with ideal boundary $\mathbb{R} \cup \{\infty\}$. Let $r \in \text{Isom}(\mathbb{H}^3)$ be the reflection through \mathcal{H} . This acts on $\mathbb{C} \cup \{\infty\}$ by complex conjugation; thus if $\mathfrak{q} \in \Gamma < \text{PSL}_2(\mathbb{C})$, then $\mathfrak{q}' = \bar{\mathfrak{q}}$, where $\bar{\mathfrak{q}} \in \text{PSL}_2(\mathbb{C})$ is the element whose entries are the conjugates of the entries of \mathfrak{q} . We will thus let $\overline{\Gamma}$ denote Γ^r , the quotient of Γ by r .

Lemma 3.3. *Define $c = \begin{pmatrix} 1 & i\sqrt{2} \\ 0 & 1 \end{pmatrix}$, and let $\Gamma_T = \langle \Gamma_{T_0}, \overline{\Gamma_{T_0}}^{c^{-2}} \rangle$. There is an inclusion-induced isomorphism*

$$\Gamma_{T_0} *_{\Lambda^k} \overline{\Gamma_{T_0}}^{c^{-2}} \rightarrow \Gamma_T,$$

and an isometry $C(\Gamma_{T_0}) \cup_{F'} C(\overline{\Gamma_{T_0}}^{c^{-2}}) \rightarrow C(\Gamma_T)$ determined by the natural maps. Furthermore, $c^{-2}r$ normalizes Γ_T , and $\phi_{c^{-2}r}: C(\Gamma_T) \rightarrow C(\Gamma_T)$ exchanges complementary components of F' .

Proof. Recall from Lemmas 2.4 and 2.6 that Λ and Λ^k are the stabilizers in Γ_{T_0} of the geodesic planes \mathcal{H} and $k(\mathcal{H})$, respectively, and that these planes project to the components of $\partial C(\Gamma_{T_0})$. It follows that \mathcal{H} and $k(\mathcal{H})$ are components of the boundary of $\text{Hull}(\Gamma_{T_0})$, hence that $\text{Hull}(\Gamma_{T_0})$ is contained in the region between them.

With \mathbf{c} as defined in the statement of the lemma, we note that $\mathbf{c}(\mathcal{H})$ has ideal boundary $\mathbb{R} + i\sqrt{2}\cup\{\infty\}$, and that $\text{ck}(\mathcal{H}) = \mathcal{H}$. Since $\text{Hull}(\Gamma_{T_0}^{\mathbf{c}})$ has boundary components $\mathbf{c}(\mathcal{H})$ and $\text{ck}(\mathcal{H}) = \mathcal{H}$, and $\Lambda^{\text{ck}} = \text{Stab}_{\Gamma_{T_0}^{\mathbf{c}}}(\mathcal{H})$ is invariant under conjugation by r , $\Gamma_{T_0}^{\mathbf{c}}$ and $\overline{\Gamma_{T_0}^{\mathbf{c}}}$ meet cute along \mathcal{H} .

Applying Lemma 3.2, we obtain an isomorphism $\Gamma_{T_0}^{\mathbf{c}} *_{\Lambda^{\text{ck}}} \overline{\Gamma_{T_0}^{\mathbf{c}}} \rightarrow \langle \Gamma_{T_0}^{\mathbf{c}}, \overline{\Gamma_{T_0}^{\mathbf{c}}} \rangle$ and an isometry

$$C(\Gamma_{T_0}^{\mathbf{c}}) \cup_{\phi_{\mathbf{c}}(F')} C(\overline{\Gamma_{T_0}^{\mathbf{c}}}) \rightarrow C(\langle \Gamma_{T_0}^{\mathbf{c}}, \overline{\Gamma_{T_0}^{\mathbf{c}}} \rangle)$$

induced by the natural maps. It is clear that r normalizes $\langle \Gamma_{T_0}^{\mathbf{c}}, \overline{\Gamma_{T_0}^{\mathbf{c}}} \rangle$, exchanging amalgamands, hence ϕ_r acts as an orientation-reversing involution of $C(\langle \Gamma_{T_0}^{\mathbf{c}}, \overline{\Gamma_{T_0}^{\mathbf{c}}} \rangle)$, fixing F' and exchanging $C(\Gamma_{T_0}^{\mathbf{c}})$ with $C(\overline{\Gamma_{T_0}^{\mathbf{c}}})$.

Observe that $\bar{\mathbf{c}} = \mathbf{c}^{-1}$. It follows that $\overline{\Gamma_{T_0}^{\mathbf{c}}} = \overline{\Gamma_{T_0}^{\mathbf{c}^{-1}}}$, and hence that $\Gamma_T = \langle \Gamma_{T_0}^{\mathbf{c}}, \overline{\Gamma_{T_0}^{\mathbf{c}}} \rangle^{\mathbf{c}^{-1}}$. Conjugating the groups of the paragraph above by \mathbf{c}^{-1} , we obtain an inclusion-induced isomorphism $\Gamma_{T_0} *_{\Lambda^{\text{ck}}} \overline{\Gamma_{T_0}^{\mathbf{c}^{-2}}} \rightarrow \Gamma_T$, and an isometry

$$C(\Gamma_{T_0}) \cup_{F'} C(\overline{\Gamma_{T_0}^{\mathbf{c}^{-2}}}) \rightarrow C(\Gamma_T)$$

induced by the natural maps. Furthermore, $\mathbf{c}^{-1}r\mathbf{c} = \mathbf{c}^{-2}r$ normalizes Γ_T and induces an orientation-reversing involution $\phi_{\mathbf{c}^{-2}r}$ fixing F' and exchanging its sides. \square

The advantage that Γ_T has over Γ_{T_0} for our purposes is that the components $\partial C(\Gamma_T)$ are exchanged by the ‘‘doubling involution’’ $\phi_{\mathbf{c}^{-2}r}$. In particular, they are naturally orientation-reversing isometric. Recall from Lemma 2.4 that $\Lambda = \text{Stab}_{\Gamma_{T_0}}(\mathcal{H})$; thus by Lemma 3.3, $\Lambda = \text{Stab}_{\Gamma_T}(\mathcal{H})$. We will again refer by $i_{\pm}^{(0)}$ to the natural map $F^{(0)} \rightarrow C(\Gamma_T)$. Then the lemma below follows from Lemma 3.3.

Lemma 3.4. *Let $F^{(1)} = \Lambda^{\mathbf{c}^{-2}}$, and let $\phi_{\mathbf{c}^{-2}}: F^{(0)} \rightarrow F^{(1)}$ and $\iota_{\pm}^{(1)}: F^{(1)} \rightarrow C(\Gamma_T)$ be the natural maps. Then $\partial C(\Gamma_T) = \partial_- C(\Gamma_T) \sqcup \partial_+ C(\Gamma_T)$, where $\partial_- C(\Gamma_T) \doteq \iota_{+}^{(0)}(F^{(0)})$ and $\partial_+ C(\Gamma_T) \doteq \iota_{-}^{(1)}(F^{(1)})$, and $\iota_{-}^{(1)}\phi_{\mathbf{c}^{-2}} = \phi_{\mathbf{c}^{-2}r}\iota_{+}^{(0)}$.*

$C(\Gamma_T)$ is a geometric model for the double of $(S^2 \times I, T_0)$ across $(S^2 \times \{1\}, \partial_+ T_0)$. Note that the double of $S^2 \times I$ across $S^2 \times \{1\}$ is again homeomorphic to $S^2 \times I$, by a map taking $(p, t) \in S^2 \times I$ to $(p, t/2)$ and $(p, t) \in \overline{S^2 \times I}$ to $(p, 1 - t/2)$.

Definition 3.5. Let $(S^2 \times I, T)$ be the double of $(S^2 \times I, T_0)$ across $(S^2 \times \{1\}, \partial_+ T_0)$. We will identify $(S^2 \times I, T_0) \subset (S^2 \times I, T)$ with its image under the map discussed above, so that $T_0 = T \cap S^2 \times [0, 1/2]$. In particular, we have $\partial_- T = \partial_- T_0 = T \cap S^2 \times \{0\}$ and $\partial_+ T_0 = T \cap S^2 \times \{1/2\}$, and we will take $\partial_+ T = T \cap S^2 \times \{1\}$.

The tangle $(S^2 \times I, T)$ is pictured in Figure 6, with $T_0 \subset T$ visible to the left of the grey vertical line representing $S^2 \times \{1/2\}$. The mirror symmetry fixing $(S^2 \times \{1/2\}, \partial_+ T_0)$,

$$r_T: (S^2 \times I, T) \rightarrow (S^2 \times I, T),$$

given by $r_T(p, x) = (p, 1 - x)$, is visible in the figure as reflection through the grey vertical line.

Proposition 3.6. *There is a homeomorphism $f_T: S^2 \times I - T \rightarrow C(\Gamma_T)$, which restricts on $S^2 \times [0, 1/2] - T_0$ to f_{T_0} followed by the natural map, such that the*

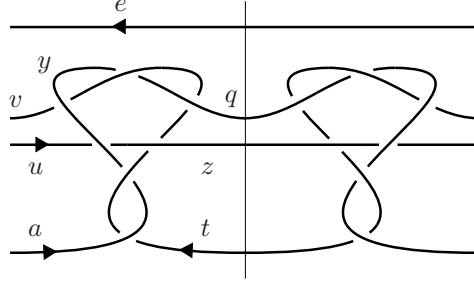


FIGURE 6. The tangle $T \subset S^2 \times I$ with labeled Wirtinger generators for T_0

diagram below commutes.

$$\begin{array}{ccc} S^2 \times I - T & \xrightarrow{f_T} & C(\Gamma_T) \\ r_T \downarrow & & \downarrow \phi_{c-2r} \\ S^2 \times I - T & \xrightarrow{f_T} & C(\Gamma_T) \end{array}$$

Furthermore, f_T takes $S^2 \times \{0\} - \partial_- T$ to $\partial_- C(\Gamma_T)$ and $S^2 \times \{1\} - \partial_+ T$ to $\partial_+ C(\Gamma_T)$.

Proof. We define f_T using the properties described in the statement of the lemma. Namely, we first require f_T to restrict on $S^2 \times [0, 1/2] - T_0$ to the homeomorphism f_{T_0} defined in Proposition 2.9, followed by the natural map $C(\Gamma_{T_0}) \rightarrow C(\Gamma_T)$. For $x \in S^2 \times [1/2, 1] - T$, we define $f_T(x) = \phi_{c-2r} f_T r_T(x)$. The resulting map is well-defined, since r_T fixes $S^2 \times \{1/2\} - \partial_+ T_0$ and ϕ_{c-2r} fixes F' . It is a homeomorphism, since r_T , f_{T_0} , and ϕ_{c-2r} are. By Corollary 2.10, f_{T_0} takes $S^2 \times \{0\} - \partial_- T_0$ to $\partial_- C(\Gamma_{T_0})$; it therefore follows from the definitions and Lemma 3.4 that $f_T(S^2 \times \{0\} - \partial_- T) = \partial_- C(\Gamma_T)$. The conclusion thus follows from the reflection equivariance of f_T . \square

The tangle T is easier to use than T_0 in a scalable construction because the totally geodesic boundary components of its geometric model $C(\Gamma_T)$ are orientation-reversing isometric. The definitions below outline this construction.

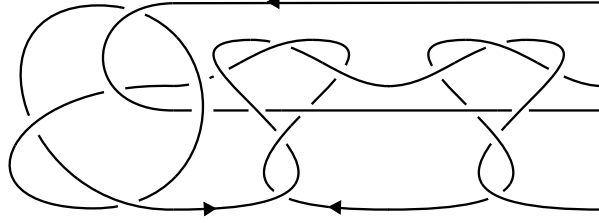
Definitions 3.7. (1) Let $j: (\partial B^3, \partial S) \rightarrow (S^2 \times \{0\}, \partial_- T)$ be the homeomorphism such that $(B^3, S) \cup_j (S^2 \times I, T)$ is the tangle pictured in Figure 7. We also refer by j to $\partial B^3 - \partial S$.

- (2) For each $i \in \mathbb{Z}$, define $h_i: S^2 \times \mathbb{R} \rightarrow S^2 \times \mathbb{R}$ by $h_i(p, x) = (p, x + (i - 1))$, and with $T \subset S^2 \times I \subset S^2 \times \mathbb{R}$, let $T^{(i)} = h_i(T)$. Now for $n \in \mathbb{N}$, define

$$(S^3, L_n) = (B^3, S) \cup_j \left(S^2 \times [0, n], \cup_{i=1}^n T^{(i)} \right) \cup_{j_n} (\overline{B^3}, \overline{S}).$$

Above, $(\overline{B^3}, \overline{S}) = r_S(B^3, S)$, where r_S is the reflective involution of S^3 fixing the boundary of an embedding of B^3 and exchanging its sides, and $j_n = r_S j^{-1} r_T h_n^{-1}: (S^{(n)}, \partial_+ T^{(n)}) \rightarrow (\partial \overline{B^3}, \partial \overline{S})$.

- (3) For $i \in \{0, 1, \dots, n\}$, let $S^{(i)}$ be the image in (S^3, L_n) of $S^2 \times \{i\}$. Using Figure 6 and taking $T \subset S^2 \times I \subset S^2 \times \mathbb{R}$, label the points of $S^{(0)} \cap L_n = S^2 \times \{0\} \cap T$ by 2, 3, 4, and 1 top-to-bottom, so that for example 2 is the

FIGURE 7. $S \cup T$

terminal point of the tangle string labeled e and 1 is the initial point of the string labeled a . Label each point of $S^{(i)} \cap L_n$ by its image under h_{i+1}^{-1} .

- Remarks.* (1) With Wirtinger generators for $\pi_1(B^3 - S)$ and $\pi_1(S^2 \times I - T)$ as labeled in Figures 2 and 6, we have $j_*(a) = a$, $j_*(u) = u$, and $j_*(v) = v$.
(2) Under the natural embedding of $S^2 \times I \subset S^2 \times \mathbb{R}$ used above, we have $T = T^{(1)}$.

We now go about constructing a geometric model of $S^3 - L_n$.

- Definitions 3.8.** (1) For $i \geq 0$, let $\Lambda^{(i)} = \Lambda^{c^{-2i}}$ and $F^{(i)} = c^{-2i}(\mathcal{H})/\Lambda^{(i)}$
(2) For $i \geq 1$, let $\Gamma_T^{(i)} = \Gamma_T^{c^{-2(i-1)}}$, and define $\phi_i = \phi_{c^{-2(i-1)}} : C(\Gamma_T) \rightarrow C(\Gamma_T^{(i)})$.

The definitions above of $F^{(0)}$ and $F^{(1)}$ above agree with our previous definitions. Also, $\Gamma_T^{(1)} = \Gamma_T$, and since $\Gamma_T^{(i)} = c^{-2(i-1)}\Gamma_T c^{2(i-1)}$, Lemma 3.4 implies that

$$\Lambda^{(i-1)} = \text{Stab}_{\Gamma_T^{(i)}}(c^{-2(i-1)}(\mathcal{H})) \quad \text{and} \quad \Lambda^{(i)} = \text{Stab}_{\Gamma_T^{(i)}}(c^{-2i}(\mathcal{H})),$$

and the resulting natural maps $i_+^{(i-1)} : F^{(i-1)} \rightarrow C(\Gamma_T^{(i)})$ and $i_-^{(i)} : F^{(i)} \rightarrow C(\Gamma_T^{(i)})$ map to the components of its totally geodesic boundary.

Proposition 3.9. For $n \in \mathbb{N}$, define $M_n = C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \dots \cup C(\Gamma_T^{(n)}) \cup C(\overline{\Gamma}_S)$, using gluing maps defined as follows:

$$\begin{aligned} \iota_+^{(0)}(\iota_-^{(0)})^{-1} &: \partial C(\Gamma_S) \rightarrow \partial_- C(\Gamma_T^{(1)}) \\ \iota_+^{(i)}(\iota_-^{(i)})^{-1} &: \partial_+ C(\Gamma_T^{(i)}) \rightarrow \partial_- C(\Gamma_T^{(i+1)}) \text{ for } 1 \leq i < n \\ \phi_r \iota_-^{(0)} \phi_{c^{2n}}(\iota_-^{(n)})^{-1} &: \partial_+ C(\Gamma_T^{(n)}) \rightarrow \partial C(\overline{\Gamma}_S) \end{aligned}$$

There is a homeomorphism $h_n : S^3 - L_n \rightarrow M_n$ which restricts on $B^3 - S$ to f_S , on $S^2 \times [i-1, i] - T^{(i)}$ to $\phi_i f_T h_i^{-1}$ for $1 \leq i \leq n$, and on $\overline{B}^3 - \overline{S}$ to $\phi_r f_S r_S$.

Proof. We will use the description of h_n above as its definition. Then by Proposition 2.8 and the definitions, h_n restricts on $B^3 - S$ and $\overline{B}^3 - \overline{S}$ to homeomorphisms to $C(\Gamma_S)$ and $C(\overline{\Gamma}_S)$, respectively. By Proposition 3.6 and the definitions, for each i between 1 and n it restricts on $S^2 \times [i-1, i] - T^{(i)}$ to a homeomorphism to $C(\Gamma_T^{(i)})$. Thus in order to show that h_n is a homeomorphism, we must only show that it is well defined on the spheres $S^{(i)} - \{1, 2, 3, 4\}$ that separate these tangle complements.

We will show first that h_n is well defined on $S^{(0)} - \{1, 2, 3, 4\}$. When $i = 1$, $T^{(i)} = T$, $\Gamma_T^{(i)} = \Gamma_T$, and h_i and ϕ_i are each the identity map. Checking well-definedness in this case therefore reduces to showing that on $\partial B^3 - \partial S$, $f_T \circ j = i_+^{(0)}(i_-^{(0)})^{-1} \circ f_S$.

By their definitions above, f_S and $f_T \circ j$ induce the same isomorphism from $\pi_1(\partial B^3 - \partial S)$ to $\Lambda = \Gamma_S \cap \Gamma_T$. Recall from Lemma 2.4 and the remarks above Lemma 3.4 that the $i_{\pm}^{(0)}$ are induced by the inclusions of Λ into Γ_S and Γ_T . Therefore at the level of fundamental group, $\left(i_+^{(0)}(i_-^{(0)})^{-1} \circ f_S\right)_* = (f_T \circ j)_*$. Since any two homeomorphisms between 4-punctured spheres that induce the same map on fundamental groups are properly isotopic, we may isotope j so that f_S and $f_T j$ agree on $S^{(0)}$. The conclusion thus follows in this case.

For $1 \leq i < n$, we may use the fact that $\Gamma_T^{(i)}$ and $\Gamma_T^{(i+1)}$ are conjugates of Γ_T to obtain the following model descriptions for $\iota_+^{(i)}$ and $\iota_-^{(i)}$:

$$(2) \quad \iota_+^{(i)} = \phi_{i+1} \iota_+^{(0)} \phi_{c^{-2i}}^{-1} \quad \iota_-^{(i)} = \phi_i \iota_-^{(1)} \phi_{c^{-2(i-1)}}^{-1}$$

Here $\iota_+^{(0)}: F^{(0)} \rightarrow \partial_- C(\Gamma_T)$ and $\iota_-^{(1)}: F^{(1)} \rightarrow \partial_+ C(\Gamma_T)$ are the natural maps of Lemma 3.4. Using the reflection-invariance property described there, we thus obtain

$$(3) \quad \iota_+^{(i)} (\iota_-^{(i)})^{-1} = \phi_{i+1} \iota_+^{(0)} \left(\iota_-^{(1)} \phi_{c^{-2}} \right)^{-1} \phi_i^{-1} = \phi_{i+1} \phi_{c^{-2r}}^{-1} \phi_i^{-1}.$$

Then by the reflection-equivariance property of f_T from Proposition 3.6, we have

$$\iota_+^{(i)} (\iota_-^{(i)})^{-1} \circ \phi_i f_T h_i^{-1} = \phi_{i+1} \phi_{c^{-2r}}^{-1} f_T h_i^{-1} = \phi_{i+1} f_T r_T h_i^{-1}.$$

It follows directly from the definitions that $r_T h_i^{-1} = h_{i+1}^{-1}$ on $S^{(i)}$, whence h_n is well defined on $S^{(i)} - \{1, 2, 3, 4\}$ for $1 \leq i < n$.

To show h_n is well defined on $S^{(n)}$ requires another definition-chase, this time to check

$$\phi_r f_S r_S \circ j_n = \phi_r \iota_-^{(0)} \phi_{c^{2n}} (\iota_-^{(n)})^{-1} \circ \phi_n f_T h_n^{-1}.$$

By Definition 3.7(2), $j_n = r_S j^{-1} r_T h_n^{-1}$; therefore simplifying the left-hand side above yields $\phi_r f_S j^{-1} r_T h_n^{-1}$. On the other hand, using the model description of $\iota_-^{(n)}$ from (2), the right-hand side above simplifies to $\phi_r \iota_-^{(0)} \phi_{c^2} (\iota_-^{(1)})^{-1} f_T h_n^{-1}$. The reflection-invariance property of Lemma 3.4 and an appeal to the case $i = 0$ establish the desired equation. \square

Corollary 3.10. *For $0 \leq i < n$, refer again by $F^{(i)}$ to the image of $\iota_+^{(i)}(F^{(i)}) \subset C(\Gamma_T^{(i)})$ under inclusion into M_n , and refer by $F^{(n)}$ to the included image of $\iota_-^{(n)}(F^{(n)})$. For each i , $F^{(i)}$ is totally geodesic in M_n and $f_n(S^{(i)} - \{1, 2, 3, 4\}) = F^{(i)}$.*

This follows immediately from Proposition 3.9 and the fact that the maps $\iota_{\pm}^{(i)}$ are isometric embeddings. The proposition below describes an algebraic model for M_n .

Proposition 3.11. *For $n \in \mathbb{N}$, define $\Gamma_n = \langle \Gamma_S, \Gamma_T^{(1)}, \dots, \Gamma_T^{(n)}, \overline{\Gamma}_S^{c^{-2n}} \rangle$. There is an isometry $M_n \rightarrow \mathbb{H}^3/\Gamma_n$ which restricts on $C(\Gamma_S)$ and each $C(\Gamma_T^{(i)})$ to the natural map, and on $C(\overline{\Gamma}_S)$ to $\phi_{c^{-2n}}$ followed by the natural map.*

Proof. We first recall from Lemma 2.4 that the plane \mathcal{H} with ideal boundary $\mathbb{R} \cup \{\infty\}$ projects $\partial C(\Gamma_S)$ under the quotient map $\mathbb{H}^3 \rightarrow \mathbb{H}^3/\Gamma_S$. From this and the fact that all ideal vertices of the octahedron \mathcal{P}_1 have non-negative imaginary part, it follows that

$$\text{Hull}(\Gamma_S) \subset \{z \in \mathbb{C} \mid \Im z \geq 0\} \cup \{\infty\}.$$

Similarly, from Lemma 3.4 and the positioning of \mathcal{P}_2 we find that

$$\text{Hull}(\Gamma_T) \subset \{z \in \mathbb{C} \mid 0 \geq \Im z \geq -2\sqrt{2}\} \cup \{\infty\}.$$

Then inspecting the action of \mathfrak{c} on $\mathbb{C} \cup \{\infty\}$, we find that any point of $\text{Hull}(\Gamma_T^{(i)})$ have imaginary part between $-2(i-1)\sqrt{2}$ and $-2i\sqrt{2}$ for $i \in \mathbb{N}$ (unless the point is at ∞).

The claim below builds an inductive picture of a family of isometrically embedded, codimension-0 submanifolds of M_n with totally geodesic boundary.

Claim. For $1 \leq i \leq n$, define $\Gamma_-^{(i)} = \langle \Gamma_S, \Gamma_T^{(1)}, \dots, \Gamma_T^{(i)} \rangle$. There is an isometry $C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \dots \cup C(\Gamma_T^{(i)}) \rightarrow C(\Gamma_-^{(i)})$, where the gluing maps for the domain are as in Proposition 3.9, which restricts on $C(\Gamma_S)$ and each $C(\Gamma_T^{(j)})$, $j < i$, to the natural map. Furthermore:

- (1) $\Lambda^{(i)} = \text{Stab}_{\Gamma_-^{(i)}}(\mathfrak{c}^{-2n}(\mathcal{H}))$, and the induced natural map is an isometry $F^{(i)} \rightarrow \partial C(\Gamma_-^{(i)})$ which factors as $\iota_-^{(i)}$ followed by the natural map $C(\Gamma_T^{(i)}) \rightarrow C(\Gamma_-^{(i)})$.
- (2) $\text{Hull}(\Gamma_-^{(i)}) \subset \{z \in \mathbb{C} \mid \Im z \geq -i\sqrt{2}\} \cup \{\infty\}$.

Proof of claim. We will prove the claim by induction. If it holds for some $i < n$, then (1) and (2) above, together with the observations above the claim imply that $\Gamma_-^{(i)}$ and $\Gamma_T^{(i+1)}$ meet cute along $\mathfrak{c}^{-2i}(\mathcal{H})$. Then by Lemma 3.2, the natural maps determine an isometry $C(\Gamma_-^{(i)}) \cup C(\Gamma_T^{(i+1)}) \rightarrow C(\Gamma_-^{(i+1)})$, where by the inductive hypothesis and the observation above Proposition 3.9, the gluing map for the domain is $\iota_+^{(i)}(\iota_-^{(i)})^{-1}$ following the inverse of the natural map.

Furthermore, since $C(\Gamma_-^{(i)})$ has a unique totally geodesic boundary component, which is isometrically identified with $\partial_- C(\Gamma_T^{(i+1)})$ in the isometry to $C(\Gamma_-^{(i+1)})$ described above, the unique totally geodesic boundary component of $C(\Gamma_-^{(i+1)})$ is the isometric image of $\partial_+ C(\Gamma_T^{(i+1)})$. Therefore the observations above Proposition 3.9 imply that this boundary component is the image of $\iota_-^{(i+1)}(F^{(i+1)})$ under the natural map. Assertion (1) of the claim thus follows for $\Gamma_-^{(i+1)}$. It follows that $\text{Hull}(\Gamma_-^{(i+1)})$ is entirely on one side or the other of the boundary at infinity of $\mathfrak{c}^{-2(i+1)}(\mathcal{H})$. Since $\Gamma_T^{(i+1)} < \Gamma_-^{(i+1)}$, assertion (2) now follows.

By our definition of ‘‘natural map’’ above Lemma 3.2, the composition of the natural map $C(\Gamma_T^{(j)}) \rightarrow C(\Gamma_-^{(i)})$, with the natural map $C(\Gamma_-^{(i)}) \rightarrow C(\Gamma_-^{(i+1)})$ is itself natural, for $j \leq i$. Hence if the claim holds for $\Gamma_-^{(i)}$, $i < n$, it holds for $\Gamma_-^{(i+1)}$. The claim will therefore hold by induction if it is true in the base case $i = 1$. But this follows from the fact that Γ_S and $\Gamma_T^{(1)}$ meet cute along \mathcal{H} . This follows in turn from Lemmas 2.4 and 3.4, which establish that $\Lambda^{(0)} = \text{Stab}_{\Gamma_S}(\mathcal{H}) = \text{Stab}_{\Gamma_T}(\mathcal{H})$, and the first paragraph of the proof. \square

Using the claim, it now follows that $\Gamma_-^{(n)}$ and $\overline{\Gamma}_S^{\mathfrak{c}^{-2n}}$ meet cute along $\mathfrak{c}^{-2n}(\mathcal{H})$; hence a final application of Lemma 3.2 implies that the natural maps determine an isometry $C(\Gamma_-^{(n)}) \cup C(\overline{\Gamma}_S^{\mathfrak{c}^{-2n}}) \rightarrow C(\Gamma_n)$. Since each of $C(\Gamma_-^{(n)})$ and $C(\overline{\Gamma}_S^{\mathfrak{c}^{-2n}})$ has a unique boundary component, $C(\Gamma_n)$ is boundaryless and hence equal to \mathbb{H}^3/Γ_n . The conclusion of the proposition follows. \square

The result below follows Proposition 3.11, or, really, its proof.

Corollary 3.12. *For fixed n and $0 \leq i \leq n$, define*

$$\Gamma_-^{(i)} = \langle \Gamma_S, \Gamma_T^{(1)}, \dots, \Gamma_T^{(i)} \rangle \quad \Gamma_+^{(i)} = \langle \Gamma_T^{(i+1)}, \dots, \Gamma_T^{(n)}, \bar{\Gamma}_S^{c^{-2n}} \rangle,$$

Then $\Gamma_+^{(i)}$ and $\Gamma_-^{(i)}$ meet cusp along $c^{-2i}(\mathcal{H})$ and the natural maps determine an isometry $C(\Gamma_-^{(i)}) \cup C(\Gamma_+^{(i)}) \rightarrow \mathbb{H}^3/\Gamma_n$. The isometry of Proposition 3.11 factors through this map, so that the component of $M_n - F^{(i)}$ containing $C(\Gamma_S)$ is taken isometrically to $C(\Gamma_-^{(i)})$.

In the remainder of the paper, we will frequently take the isometry above for granted and refer to the components obtained by splitting M_n along $F^{(i)}$ by $C(\Gamma_\pm^{(i)})$.

4. INVARIANTS

4.1. Traces. If $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ is a discrete group, its trace field $\mathbb{Q}(\mathrm{tr} \Gamma)$ is obtained by adjoining to \mathbb{Q} the traces of elements of Γ . If the hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ has finite volume, Mostow rigidity implies that this is a topological invariant of M . It follows from the local rigidity theorems of Garland-Prasad that in this case the trace field is a number field; ie, a finite extension of \mathbb{Q} . The trace field is not generally an invariant of the commensurability class of M , however, and to obtain one we pass to the *invariant trace field* $k\Gamma$. This is obtained by adjoining to \mathbb{Q} the traces of squares of elements of Γ . When M is the complement of a link in a \mathbb{Z}_2 -homology sphere, its trace field and invariant trace field coincide (cf. [17]).

Proposition 4.1. *$k(\Gamma_S) = \mathbb{Q}(i)$, $k(\Gamma_T) = \mathbb{Q}(i\sqrt{2})$, and $k(M_n) = \mathbb{Q}(i, i\sqrt{2})$ for all $n \in \mathbb{N}$.*

Proof. Its definition in Corollary 2.2 immediately implies $\Gamma_S < \mathrm{PSL}_2(\mathbb{Q}(i))$. The description in Corollary 2.3, of Γ_{T_0} , and Lemma 3.3 imply that $\Gamma_T < \mathrm{PSL}_2(\mathbb{Q}(i\sqrt{2}))$. Thus $k\Gamma_S \subseteq \mathbb{Q}(i)$, and $k\Gamma_T \subseteq \mathbb{Q}(i\sqrt{2})$. Equality is established by appealing to the fact that a Kleinian group Γ with $k\Gamma \subset \mathbb{R}$ is conjugate into $\mathrm{PSL}_2(\mathbb{R})$, hence is Fuchsian (cf. [17]). To establish the claim regarding Γ_n , we note that the element c defined in Lemma 3.3 lies in $\mathrm{PSL}_2(\mathbb{Q}(i\sqrt{2}))$, and each Γ_n is contained in the group generated by Γ_S , Γ_T , and c . \square

We say $M = \mathbb{H}^3/\Gamma$ has *integral traces* if for each $\gamma \in \Gamma$, $\mathrm{tr} \gamma$ is an algebraic integer. Otherwise we say M_n has a nonintegral trace. M has integral traces if and only if all manifolds commensurable to M do as well (cf. [17]).

Proposition 4.2. *For each n , M_n has integral traces.*

Proof. As in the proposition above, this follows from the fact that each Γ_n is contained in the group generated by Γ_S , Γ_T , and c . It is easy to see that the *entries* of the generators for Γ_S and Γ_{T_0} are algebraic integers. Since c has integral entries as well, all elements of Γ_n have integral entries, hence integral traces. \square

Bass showed that if $M = \mathbb{H}^3/\Gamma$ where Γ has an element with a nonintegral trace, there are closed essential surfaces in M associated to this trace [5]. We say that such surfaces are *detected by the trace ring*. For fixed n and $1 \leq i \leq n$, closed essential surfaces in M_n can be obtained by ‘‘tubing’’ $S^{(i)}$ through $B^3 - L_-^{(i)}$. More

precisely, let \mathcal{N}_i be a regular neighborhood of $L_-^{(i)}$ in $(B^3, L_-^{(i)}) \subset (S^3, L_n)$, let $A_i = \mathcal{N}_i \cap \overline{B^3} - \mathcal{N}_i$, and let

$$\hat{S}_i = \overline{S^{(i)} - (S^{(i)} \cap \mathcal{N}_i)} \cup A_i.$$

Then \hat{S}_i is a closed surface of genus two which is incompressible in M_n . We will show below that certain mutants have nonintegral traces, and one easily finds surfaces analogous to \hat{S}_i in the mutants. It is interesting to note that although these surfaces are present in all link complements, the trace ring does not detect any closed surfaces in the M_n .

4.2. Scissors congruence and the Bloch invariant. We will prove in Proposition 4.5 below that *scissors congruence* distinguishes the commensurability class of M_m from that of M_n for $m \neq n$. Manifolds are scissors congruent if they can be cut into identical collections of hyperbolic polyhedra. A fundamental tool for proving results about the scissors congruence relation is its connection to the Bloch invariant, which allows access to deep theorems from algebraic k -theory. We focus on this connection below, referring the reader to the thorough and very readable survey [23] for a broader exposition.

Definition 4.3. Let $k \subset \mathbb{C}$ be a field. Define the *pre-Bloch group* $\mathcal{P}(k)$ to be the quotient of the free \mathbb{Z} -module on $k - \{0, 1\}$ by all instances of the following relations.

$$(4) \quad [x] - [y] + \left[\frac{y}{x} \right] - \left[\frac{1-x^{-1}}{1-y^{-1}} \right] + \left[\frac{1-x}{1-y} \right] = 0, \quad x \neq y \in k - \{0, 1\}$$

$$(5) \quad [z] = \left[1 - \frac{1}{z} \right] = \left[\frac{1}{1-z} \right] = - \left[\frac{1}{z} \right] = - \left[\frac{z}{z-1} \right] = -[1-z], \quad z \in k - \{0, 1\}$$

There is a map $\delta: \mathcal{P}(k) \rightarrow k^* \wedge k^*$ given by $[z] \mapsto 2(z \wedge (1-z))$. (Here k^* is considered a \mathbb{Z} -module with multiplication as the group operation and \mathbb{Z} -action given by $a \cdot x = x^a$, $a \in \mathbb{Z}$.) The *Bloch group* is $\mathcal{B}(k) = \ker \delta$.

Remark. If k is algebraically closed, the relation (4) above, called the *five term relation*, implies the relation (5). For instance, taking \sqrt{z} and $\sqrt{z^{-1}}$ as x and y , respectively, in (4), then interchanging their roles and summing the results yields $[z] + [1/z] = 0$.

An ideal tetrahedron T in \mathbb{H}^3 determines an element $[z] \in \mathcal{P}(\mathbb{C})$ as follows. Let z_1, z_2, z_3 , and z_4 be the ideal vertices of T on $\widehat{\mathbb{C}} = \partial_\infty \mathbb{H}^3$, ordered as follows. The intersection of T with a sufficiently small horosphere H centered at z_4 determines a Euclidean triangle in T , which we give the boundary orientation from the component of $\mathbb{H}^3 - H$ containing the remaining ideal vertices. We order them so that traversing the boundary of the triangle $H \cap T$ in the direction determined by its orientation visits the vertices determined by z_1, z_2 , and z_3 in that order (up to cyclic permutation). We define the *cross ratio parameter* of T as

$$z = [z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}.$$

The choices made above only determine the ordering on the ideal vertices of T up to even permutation. It is easily verified that changing the order by an even permutation replaces z by one of z , $\frac{1}{1-z}$, or $1 - \frac{1}{z}$. Thus by the relation (5), $[z]$ is a well-defined element of $\mathcal{P}(\mathbb{C})$. Since the cross ratio is invariant under Möbius

transformations, $[z]$ is an invariant of the orientation-preserving isometry class of T . Changing the ordering of the ideal vertices by an odd permutation replaces z by one of $\frac{1}{z}$, $\frac{z}{z-1}$, or $1-z$.

Definition 4.4. Let $M = \mathbb{H}^3/\Gamma$ be a complete, orientable hyperbolic 3-manifold of finite volume with a specified triangulation; that is, $M = T_1 \cup \dots \cup T_n$, each T_i isometric to an ideal hyperbolic tetrahedron, with $T_i \cap T_j$ either empty, an edge of each, or a face of each, for $i \neq j$. For each i in $\{1, \dots, n\}$, let z_i be the cross ratio parameter of T_i , and define the *Bloch invariant* $\beta(M)$ as follows.

$$\beta(M) = [z_1] + [z_2] + \dots + [z_n] \in \mathcal{B}(\mathbb{C})$$

If M is a complete, orientable hyperbolic 3-manifold of finite volume with totally geodesic boundary, a triangulation of M determines one of its double DM . Define $\beta(M) = \frac{1}{2}\beta(DM)$.

Remark. It is not clear *a priori* that $\delta(\beta(M)) = 0$, but this follows from a geometric interpretation of the Bloch invariant. Neumann-Yang [24] showed that $\beta(M)$ does not depend on the choice of triangulation, and furthermore that $\beta(M) \in \mathcal{B}(k)$ when M is not compact, where k is the trace field of M .

Recall that the cross ratio $z = [z_1 : z_2 : z_3 : z_4]$ is characterized by the property that z is the destination of z_1 under the Möbius transformation sending z_2, z_3 , and z_4 to $1, \infty$, and 0 , respectively. For an ideal tetrahedron T with its ideal vertices ordered as described above, it follows that the cross ratio parameter of T lies in the upper half plane.

The *Bloch-Wigner dilogarithm function* $D_2: \mathbb{C} - \{0, 1\} \rightarrow \mathbb{R}$ describes the volume of an ideal tetrahedron in terms of its cross ratio parameter. We record the formula below.

$$D_2(z) = \Im\psi(z) + \log|z| \arg(1-z), \text{ where } \psi(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^2}.$$

For z in the upper half plane, $D_2(z)$ is the volume of the ideal tetrahedron with ideal vertices $z, 1, \infty$, and 0 . Note that $D_2(\bar{z}) = -D_2(z)$. The theorem below uses D_2 to define a map known as the *Borel regulator*.

Theorem ([6]). *Suppose k is a number field with complex embeddings $\sigma_1, \dots, \sigma_{r_2}$, one representing each complex-conjugate pair. There is a map $\mathcal{B}(k) \rightarrow \mathbb{R}^{r_2}$ extending $[z] \mapsto (D_2(\sigma_1(z)), \dots, D_2(\sigma_{r_2}(z)))$, which takes $\mathcal{B}(k)$ onto a lattice in \mathbb{R}^{r_2} , with kernel consisting entirely of torsion elements.*

The octahedron \mathcal{P}_1 may be divided into a collection of 4 tetrahedra by the addition of a single edge, joining the ideal vertices $(1+i)/2$ and ∞ , and four triangular faces. Each new face has vertices $(1+i)/2, \infty$, and one of the remaining vertices of \mathcal{P}_1 . The resulting triangulation projects to a triangulation of M_S . We leave it as an exercise to the reader to triangulate the cuboctahedron \mathcal{P}_2 , with the additional requirement that the induced triangulation of its internal faces is preserved by the face pairings. Such a triangulation projects to one of M_{T_0} .

The map p_S of Corollary 2.2 takes the triangulation of M_S described above to one of $C(\Gamma_S)$, and the map p_{T_0} of Corollary 2.3 does the same for $C(\Gamma_{T_0})$. Since the external faces of \mathcal{P}_1 and \mathcal{P}_2 are ideal triangles, they are faces of any triangulation obtained by subdivision; hence the triangulations inherited by $C(\Gamma_S)$ and $C(\Gamma_{T_0})$ intersect their boundaries in Δ_S and $\Delta_{T_0}^{\pm}$, respectively (as defined below Corollaries

2.2 and 2.3). Then Lemma 3.3 implies that $C(\Gamma_T)$ is triangulated by the union of the triangulation of $C(\Gamma_{T_0})$ and its image under $\phi_{c^{-2r}}$.

Now fix $n \in \mathbb{N}$. For each i between 1 and n , $C(\Gamma_T^{(i)})$ is triangulated by the image under ϕ_i of the triangulation of $C(\Gamma_T)$. Also, $C(\overline{\Gamma_S}^{c^{-2n}})$ is triangulated by the image under $\phi_{c^{-2n}}$ of that of $C(\Gamma_S)$. By the construction in Section 3, the union of these triangulates M_n if the gluing maps $\iota_+^{(i)}(\iota_-^{(i)})^{-1}$ preserve their boundaries. But this follows from assertion (3) of Lemma 2.4 and from Lemma 3.4. Hence M_n is triangulated in this fashion. Below, the details of the triangulation of M_n are not so important as this fact.

Proposition 4.5. *For $m \neq n$, M_m is not commensurable with M_n .*

Proof. Let β_1 be the Bloch invariant of $C(\Gamma_S)$, and let β_2 be the Bloch invariant of $C(\Gamma_{T_0})$. Then $2 \cdot \beta_1 \in \mathcal{B}(\mathbb{Q}(i))$, and applying the Borel regulator map to $2 \cdot \beta_1$ yields the volume of the double of $C(\Gamma_S)$, which is twice its volume v_1 . Since $C(\Gamma_T)$ is itself a double, the Borel regulator takes $\beta_2 \in \mathcal{B}(\mathbb{Q}(i\sqrt{2}))$ to the volume v_2 of $C(\Gamma_T)$. It follows that β_1 and β_2 have infinite order in the pre-Bloch group.

The inclusion of $\mathbb{Q}(i)$ into $k = \mathbb{Q}(i, i\sqrt{2})$ induces a natural map $\mathcal{B}(\mathbb{Q}(i)) \rightarrow \mathcal{B}(k)$, and similarly for $\mathcal{Q}(i\sqrt{2}) \subset k$. It is clear that k has a pair of complex conjugate embeddings, each determined by its action on i and $i\sqrt{2}$. We will take $\sigma_1 = id_k$, and $\sigma_2(i) = i$, $\sigma_2(i\sqrt{2}) = -i\sqrt{2}$, in defining the Borel regulator on k . Since each σ_i restricts on $\mathbb{Q}(i)$ to the identity, the Borel regulator on k takes β_1 to (v_1, v_1) . On the other hand, the Borel regulator on k takes β_2 to $(v_2, -v_2)$. Thus β_1 and β_2 are linearly independent in $\mathcal{B}(k)$, since their images are linearly independent in \mathbb{R}^2 .

For any n , the triangulation which M_n inherits, from the two copies of $C(\Gamma_S)$ and n copies of $C(\Gamma_T)$ which comprise it, has Bloch invariant $2\beta_1 + 2n\beta_2 \in \mathcal{B}(k)$. If $\widetilde{M} \rightarrow M_n$ is a covering space of degree k , the preimage in \widetilde{M} of each tetrahedron T in the triangulation of M_n is a disjoint union of k isometric copies of T . Thus $\beta(\widetilde{M}) = k\beta(M_n)$. If M_m were commensurable with M_n , it would follow that there is a rational number q such that

$$q[2\beta_1 + 2m\beta_2] = 2\beta_1 + 2n\beta_2.$$

But since β_1 and β_2 are linearly independent, this forces $q = 1$ and $m = n$. \square

4.3. Cusp parameters.

Definition 4.6. Let $T = \mathbb{C}/\Lambda$ be a Euclidean torus, where $\Lambda \subset \mathbb{C}$ is a lattice. Define the *complex modulus* of T as $m(T) = \alpha/\beta$, where $\Lambda = \langle \alpha, \beta \rangle$.

Remark. This is well defined only up to the action of $\mathrm{PGL}_2(\mathbb{Z})$ by Möbius transformations: a different generating pair $\gamma = p\alpha + q\beta$, $\delta = r\alpha + s\beta$ determines an invertible matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Computing the modulus with γ and δ yields

$$m(T) = \frac{p\alpha + q\beta}{r\alpha + s\beta} = \frac{p(\alpha/\beta) + q}{r(\alpha/\beta) + s}.$$

It will prove useful here to understand the complex modulus of a torus by decomposing it into annuli using a family of parallel geodesics.

Definition 4.7. For a Euclidean annulus A with core of length ℓ and distance d between geodesic boundary components, define the *real modulus* of A by $m(A) = d/\ell$.

If $T = \mathbb{C}/\Lambda$, and $\Lambda = \langle \alpha, \beta \rangle$, then α and β determine simple closed geodesics on T intersecting once. These are the projections to T of the line segments in \mathbb{C} joining 0 to α and β , respectively. Below let A_β denote the Euclidean annulus with geodesic boundary obtained as the path completion of the metric on $T - \beta$ inherited from T .

Lemma 4.8. *Let $T = \mathbb{C}/\Lambda$ be a Euclidean torus, and suppose α, β is a generating pair for Λ . Decompose $m(T)$ into real and imaginary parts:*

$$m(T) = \tau_\beta + i \cdot \mu_\beta,$$

where $\tau_\beta = \Re(\alpha/\beta)$ and $\mu_\beta = \Im(\alpha/\beta) \in \mathbb{R}$. Then $\tau_\beta = \frac{\|\alpha\|}{\|\beta\|} \cos \theta$, where θ is the angle between the geodesics α and β on T , and $|\mu_\beta| = m(A_\beta)$.

Proof. Write $\alpha = \|\alpha\|e^{i\theta_1}$ and $\beta = \|\beta\|e^{i\theta_2}$. Then $\theta = \theta_1 - \theta_2$ is the angle between the geodesics corresponding to α and β , and $\frac{\alpha}{\beta} = \frac{\|\alpha\|}{\|\beta\|}e^{i\theta}$. Writing $e^{i\theta} = \cos \theta + i \sin \theta$ yields the first assertion immediately.

To establish the second, consider the strip \tilde{A}_β in \mathbb{C} bounded by the line containing 0 and β and its translate by α , containing α and $\alpha + \beta$. The quotient of \tilde{A}_β induced by the action of β is the universal covering $\tilde{A}_\beta \rightarrow A_\beta$. The distance between boundary components of \tilde{A}_β is $\|\alpha\| |\sin \theta|$, and the length of the core of A_β is the translation length of β , which is $\|\beta\|$. \square

Lemma 4.8 provides a convenient means for understanding the modulus of a Euclidean torus in terms of ‘‘Fenchel-Nielsen’’ coordinates (μ_β, τ_β) associated to a simple closed geodesic β . We regard μ_β as a sort of length parameter for the annulus A_β , and τ_β as a twist parameter.

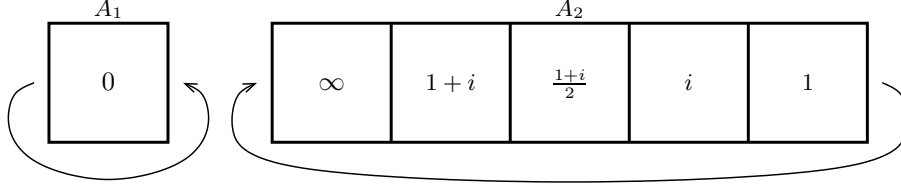
Lemma 4.9. *Suppose T is a Euclidean torus decomposed into annuli A_1, \dots, A_n by simple closed geodesics parallel to β . Then*

$$|\mu_\beta| = m(A_1) + m(A_2) + \dots + m(A_n).$$

Proof. By isotoping β if necessary, we may assume that it is one of the geodesics determining the A_i ; hence $A_\beta = A_1 \cup A_2 \dots \cup A_n$. Then if α_0 is an arc perpendicular to ∂A_β , joining one component to the other, for each i , $\alpha_0 \cap A_i$ is an arc perpendicular to ∂A_i joining one component to the other. This is because ∂A_i is parallel to β . Since $\ell(\alpha_0) = \sum_i \ell(\alpha_0 \cap A_i)$ and the core of each A_i has length $\ell(\beta)$, the result follows. \square

The annuli we are concerned with arise as horospherical cross sections of the cusps of M_S and M_T . Recall from Lemma 2.4 that $\text{Stab}_{\Gamma_S}(\mathcal{H})$ is a group Λ generated by parabolic isometries $\mathfrak{p}_1, \mathfrak{p}_2$ and \mathfrak{p}_3 . Furthermore, as pointed out in Remark 1 below the lemma, \mathfrak{p}_1 and \mathfrak{p}_3 are conjugate in Γ_S , as are \mathfrak{p}_2 and $\mathfrak{p}_4 = \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3^{-1}$. We asserted there that $C(\Gamma_S)$ has two cusps, one corresponding to \mathfrak{p}_1 and one to \mathfrak{p}_2 . This follows from the lemma below.

In what follows, we let $\mathcal{V}_1 = \{\infty, 0, 1, i, 1+i, (1+i)/2\}$, the set of ideal vertices of the ideal octahedron \mathcal{P}_1 . Let $\{h_v \mid v \in \mathcal{V}_1\}$ be a collection of horospheres invariant under the action of the symmetry group of \mathcal{P}_1 , such that h_v is centered at v for each $v \in \mathcal{V}$ and h_∞ is at height 2.

FIGURE 8. Cross sections of the cusps of M_S

Lemma 4.10. *The projection to $M_{\mathcal{P}_1}$ of $\bigcup (h_v \cap \mathcal{P}_1)$ is a disjoint union of Euclidean annuli A_1 and A_2 with geodesic boundary, such that $p_S(A_1)$ is a horospherical cross section of the cusp of $C(\Gamma_S)$ corresponding to \mathfrak{p}_1 , $p_S(A_2)$ is a cross section of the cusp corresponding to \mathfrak{p}_2 , and $m(A_1) = 1$, $m(A_2) = 1/5$.*

Proof. Since h_∞ is at height 2 and our embedding of \mathcal{P}_1 is as in Figure 2, $h_\infty \cap \mathcal{P}_1$ is a square with sides of length $1/2$. Since the symmetry group of \mathcal{P}_1 acts transitively on vertices, this holds for all $h_v \cap \mathcal{P}_1$, $v \in \mathcal{V}$. We will call a side of $h_v \cap \mathcal{P}_1$ *internal* if it is contained in an internal face of \mathcal{P}_1 and *external* otherwise. The face-pairing \mathfrak{s} has the property that if v and v' are ideal vertices of \mathcal{P}_1 and $\mathfrak{s}(v) = v'$, then $\mathfrak{s}(h_v) = h_{v'}$, and $\mathfrak{s}(h_v \cap \mathcal{P}_1)$ abuts $h_{v'} \cap \mathcal{P}_1$ along an internal side. The analogous property holds for \mathfrak{t} .

Each of \mathfrak{s} and \mathfrak{t} identifies a pair of internal faces of \mathcal{P}_1 , yielding M_S . The isometry p_S of Corollary 2.2 is induced by the inclusion $\mathcal{P}_1 \rightarrow \mathbb{H}^3$. Since $\mathfrak{p}_1 = \mathfrak{s}^{-1}$ fixes the ideal vertex of \mathcal{P}_1 at 0, it identifies the opposite internal sides of $h_0 \cap \mathcal{P}_1$. This square thus projects to a cross section of the cusp of M_S mapping under p_S to the cusp of $C(\Gamma_S)$ corresponding to \mathfrak{p}_1 ; that is, A_1 . This is depicted on the left side of Figure 8.

The other cusp cross section of M_S , the annulus A_2 , is the identification space of the collection

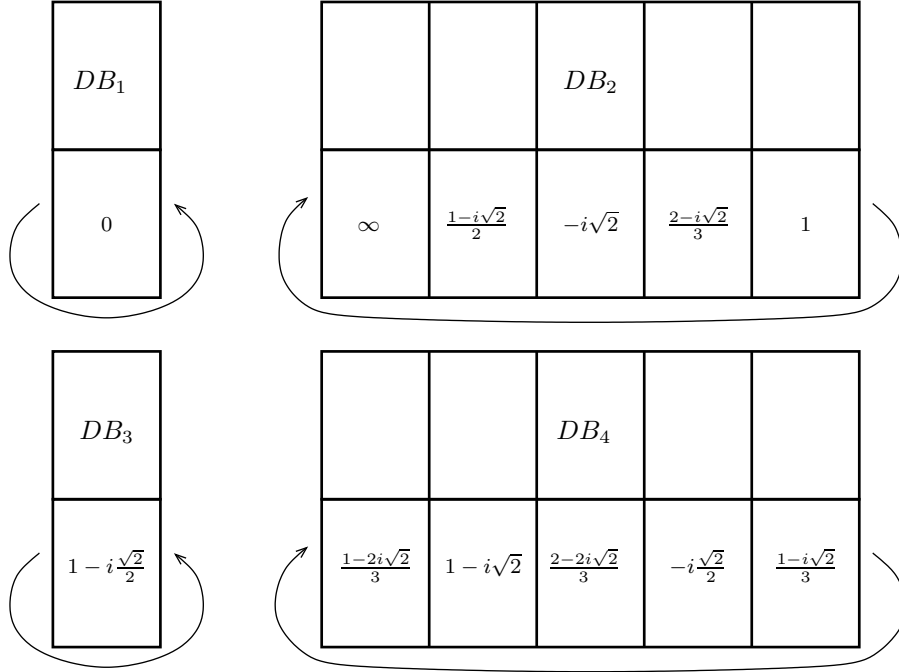
$$\{h_v \cap \mathcal{P}_1 \mid v \in \mathcal{V} - \{0\}\}$$

shown on the right side of Figure 8. In the figure, each square is the projection to M_S of $h_v \cap \mathcal{P}_1$ for the ideal vertex v by which it is labeled. The combinatorics can be verified by considering the action of \mathfrak{s} and \mathfrak{t} on \mathcal{V} .

By assumption each square in Figure 8 has side length $1/2$, and so the cores of A_1 and A_2 have lengths $1/2$ and $5/2$, respectively. For any square in Figure 8, a vertical side projects to an arc joining the distinct boundary components of the corresponding A_i , hence the distance between them is $1/2$. Thus it follows directly from the definition that $m(A_1) = 1$ and $m(A_2) = 1/5$. \square

The lemma below describes the moduli of the cusps of $C(\Gamma_{T_0})$. We asserted below Lemma 2.6 that $C(\Gamma_{T_0})$ has four cusps, one corresponding to each \mathfrak{p}_i , $i \in \{1, 2, 3, 4\}$, and each joining $\partial_- C(\Gamma_{T_0})$ to $\partial_+ C(\Gamma_{T_0})$. This follows from Lemma 4.11 below. Let \mathcal{V}_2 be the set of ideal vertices of \mathcal{P}_2 , and consider a collection of horospheres $\{h_v \mid v \in \mathcal{V}_2\}$, invariant under the symmetry group of \mathcal{P}_2 , such that h_v is centered at v for each $v \in \mathcal{V}$ and h_∞ is at height 2.

Lemma 4.11. *The projection of $\bigcup (h_v \cap \mathcal{P}_2)$ to M_{T_0} is a collection of disjoint Euclidean annuli B_j with geodesic boundary, $j \in \{1, 2, 3, 4\}$, such that $p_{T_0}(B_j)$ is a*

FIGURE 9. Cross sections of the cusps of M_T

cross section of the cusp of $C(\Gamma_{T_0})$ corresponding to $\mathfrak{p}_j \in \Lambda$, and $m(B_1) = m(B_3) = \sqrt{2}$, $m(B_2) = m(B_4) = \sqrt{2}/5$.

Proof. For $v \in \mathcal{V}_2$, we again call a side of $h_v \cap \mathcal{P}_2$ external if it is contained in an external face of \mathcal{P}_2 and internal otherwise. Each cusp cross-section of M_{T_0} is the projection of a subcollection of the $h_v \cap \mathcal{P}_2$, identified along their internal faces. From Figure 4, we find that $h_\infty \cap \mathcal{P}_2$ is a Euclidean rectangle with two opposite internal sides and two external. Since the symmetry group of \mathcal{P}_2 is transitive on its set of ideal vertices, this holds for the other h_v as well. It follows that each cusp cross section of M_{T_0} is a Euclidean annulus with geodesic boundary.

In Figure 9, the lower rectangles of each annulus DB_j are labeled by vertices v such that $h_v \cap \mathcal{P}_2$ projects to a subrectangle of cross-section of the cusp of M_{T_0} whose image under p_{T_0} corresponds \mathfrak{p}_j . Then B_j is the lower half of DB_j . We will justify this below.

The isometries \mathfrak{f} , \mathfrak{g} , and \mathfrak{h} defined in Corollary 2.3 identify the internal sides of \mathcal{P}_2 in pairs, yielding the manifold M_{T_0} with totally geodesic boundary. The parabolic $\mathfrak{p}_1 = \mathfrak{f}^{-1}$ fixes 0, identifying the internal sides of \mathcal{P}_2 sharing this ideal vertex. Thus in M_{T_0} , B_1 consists of $h_0 \cap \mathcal{P}_2$ with its internal sides identified. The description of the \mathfrak{p}_i in terms of \mathfrak{f} , \mathfrak{g} , and \mathfrak{h} above Lemma 2.4 shows that \mathfrak{p}_3 is a conjugate of \mathfrak{g}^{-1} . Since \mathfrak{g}^{-1} fixes $1 - i\frac{\sqrt{2}}{2}$, identifying the internal edges of \mathcal{P}_2 which abut it, $h_{1-i\frac{\sqrt{2}}{2}}$ projects to B_3 in M_{T_0} . This justifies the depictions of B_1 and B_3 in Figure 9.

Since \mathfrak{p}_2 fixes ∞ , $h_\infty \cap \mathcal{P}_2$ projects to a subrectangle of B_2 . Since \mathfrak{g} takes the internal side Y_3 to Y'_1 and ∞ to $(1 - i\frac{\sqrt{2}}{2})/2$, in B_2 the projection of $h_\infty \cap \mathcal{P}_2$ meets

the projection of $h_{(1-i\sqrt{2})/2} \cap \mathcal{P}_2$ along a side contained in the projection of Y_3 to M_{T_0} . Since the internal face of \mathcal{P}_2 meeting Y_1' at $(1-i\sqrt{2})/2$ is Y_3' , and this is taken to Y_2' by h^{-1} , the rectangle meeting the projection of $h_{(1-i\sqrt{2})/2}$ in B_2 on the internal side opposite its intersection with h_∞ is $h_{-i\sqrt{2}}$. Further consideration of the combinatorics of the action of f , g , and h yields the depictions of B_2 and B_4 in the figure.

From Figure 4, we find that the internal sides of $h_\infty \cap \mathcal{P}_2$ have length $\sqrt{2}/2$ and the external sides length $1/2$. Since the symmetry group of \mathcal{P}_2 is transitive on its ideal vertices, the same holds for each rectangle $h_v \cap \mathcal{P}_2$. Thus the cores of B_1 and B_3 have length $1/2$, and the cores of B_2 and B_4 have length $5/2$. For any square $h_v \cap \mathcal{P}_2$, an internal side projects to a perpendicular arc joining opposite sides of the cusp cross-section in M_{T_0} containing $h_v \cap \mathcal{P}_2$. The moduli are thus as described. \square

Recall that Lemma 3.3 describes $C(\Gamma_T)$ as the double of $C(\Gamma_{T_0})$ across the image of $F' = k(\mathcal{H})/\Lambda^k$ under the natural map described in Lemma 2.6. Therefore p_{T_0} determines a reflection-invariant map from the double M_T of M_{T_0} across $\partial_+ M_{T_0}$ to $C(\Gamma_T)$. Furthermore, as we remarked below Lemma 2.6, each cusp of $C(\Gamma_{T_0})$ joins one component of $\partial C(\Gamma_{T_0})$ to the other. Therefore taking $DB_j \subset M_T$, $j \in \{1, 2, 3, 4\}$, to be the double of B_j across its component of intersection with $\partial_+ M_{T_0}$, we have:

Lemma 4.12. *For each $j \in \{1, 2, 3, 4\}$, the image in $C(\Gamma_T)$ of DB_j is a cross section of the cusp corresponding to \mathfrak{p}_j , and $m(DB_1) = m(DB_3) = 2\sqrt{2}$, $m(DB_2) = m(DB_4) = 2\sqrt{2}/5$.*

The complex modulus is clearly a similarity invariant of a Euclidean torus. We will say T and T' are *commensurable* if T has a finite cover which is similar to a cover of T' . The following lemma can be verified by comparing the modulus of a lattice in \mathbb{C}^2 to moduli of finite-index sublattices.

Lemma 4.13. *If T and T' are commensurable, their moduli differ by the action of an element of $\mathrm{PGL}_2(\mathbb{Q})$.*

It is a well known consequence of Margulis' lemma that each cusp C of a hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ of finite volume is foliated by similar Euclidean tori, the projections to M of horospheres in \mathbb{H}^3 centered at the fixed point of a parabolic subgroup of Γ corresponding to C .

Definition 4.14. The *parameter* of a cusp C of a finite-volume complete hyperbolic manifold is the complex modulus of a horospherical cross-section of C .

Below we describe the cusp parameters of the M_n up to the action of $\mathrm{PGL}_2(\mathbb{Q})$.

Proposition 4.15. *Let T_1 be a cusp cross section of M_n such that $T_1 \cap M_S = A_1$, and let T_2 be a cross section of the other cusp of M_n . Then $m(T_1) = i(2 + 4n\sqrt{2})$, and $m(T_2) = i(2 + 4n\sqrt{2})/5$.*

Remark. These parameters are identical up to the action of $\mathrm{PGL}_2(\mathbb{Q})$, but distinct under the action of $\mathrm{PGL}_2(\mathbb{Z})$.

Proof. For i between 1 and n and $j \in \{1, 2, 3, 4\}$, we define $DB_j^{(i)} = \phi_i \circ p_T(DB_j)$, a cusp cross section in $C(H^{(i)})$, and let $\partial_\pm B_j^{(i)} = DB_j^{(i)} \cap \partial_\pm C(H^{(i)})$. We will also refer by $DB_j^{(i)}$ to its image in $C(\Gamma_n)$ under the natural map.

Recall from Proposition 3.9 that $M_n = C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \dots \cup C(\Gamma_T^{(n)}) \cup C(\overline{\Gamma}_S)$, where the gluing maps joining convex cores in the domain factor through the inclusion-induced isometries $\iota_{\pm}^{(i)}$ defined on $F^{(i)}$, $0 \leq i \leq n$.

By Lemma 4.10 above, $p_S(A_1)$ is a cross section of the cusp of $C(\Gamma_S)$ corresponding to \mathfrak{p}_1 , and by Lemma 4.12, $p_T(DB_1)$ is a cross-section of the cusp of $C(\Gamma_T)$ corresponding to \mathfrak{p}_1 . It thus follows from Lemma 2.4 that $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ takes a component of ∂A_1 to $\partial_- DB_1$. In Remark 1 below that lemma, we note that \mathfrak{p}_1 and \mathfrak{p}_3 are conjugate in Γ_S ; therefore the other component of ∂A_1 must be a cross section of the cusp of $\partial C(\Gamma_S)$ corresponding to \mathfrak{p}_3 . Hence $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ carries this component of ∂A_1 to $\partial_- DB_3$.

The definitions and Lemma 3.4 imply that for each i between 1 and $n-1$, $\iota_+^{(i)}(\iota_-^{(i)})^{-1}$ takes $\partial_+ DB_j^{(i)}$ to $\partial_- DB_j^{(i+1)}$. This uses the fact that DB_j is by its construction invariant under the doubling involution ϕ_{c-2r} , and the characterization (3) from the proof of Proposition 3.9:

$$\iota_+^{(i)}(\iota_-^{(i)})^{-1} = \phi_{i+1}\iota_+^{(0)}\phi_{c-2}^{-1} \left(\iota_-^{(1)} \right)^{-1} \phi_i^{-1} = \phi_{i+1}\phi_{c-2r}\phi_i^{-1}.$$

The above implies that T_1 is decomposed by its intersection in M_n with the separating spheres $F^{(i)}$ into the following collection of Euclidean annuli with geodesic boundary.

$$A_1 \cup DB_1^{(1)} \cup \dots \cup DB_1^{(n)} \cup \overline{A}_1 \cup DB_3^{(1)} \cup \dots \cup DB_3^{(n)}$$

Similarly, we find that T_2 decomposes into the union of A_2 , \overline{A}_2 , and $DB_j^{(i)}$ for $1 \leq i \leq n$ and $j = 2, 4$. We may take β_1 to be the geodesic $\partial_- B_1^{(1)}$ on T_1 and $\beta_2 = \partial_- B_2^{(1)} \subset T_2$. Then we obtain the following from Lemma 4.9, applying lemmas 4.10 and 4.12.

$$\Im(m(T_1)) = \pm(2 + 4n\sqrt{2}) \quad \Im(m(T_2)) = \pm \frac{2 + 4n\sqrt{2}}{5}$$

We will show $m(T_1)$ and $m(T_2)$ have real part equal to 0 by describing geodesics α_j , $j = 1, 2$, which meet the β_j once, perpendicularly. Let a_1 be the arc in A_1 which is the projection of the internal edges of $h_0 \cap \mathcal{P}_1$, the vertical arcs on the left-hand square in Figure 8. Recall that the internal edges of $h_0 \cap \mathcal{P}$ are its intersection with internal faces of \mathcal{P}_1 ; in particular, ∂a_1 is the intersection of A_1 with the one-skeleton of the triangulation Δ_S defined below Corollary 2.2.

Let $b_1 \subset B_1$ and $b_3 \subset B_3$ similarly be projections of internal edges of $h_0 \cap \mathcal{P}_2$ and $h_{1-i\sqrt{2}/2} \cap \mathcal{P}_2$, respectively (see Figure 9), and let db_1 and db_3 be the geodesic arcs of DB_1 and DB_3 containing them. Let $db_j^{(i)} = \phi_i \circ p_T(db_j)$, and let $\partial_{\pm} db_j^{(i)} = db_j^{(i)} \cap \partial_{\pm} DB_j^{(i)}$, $j = 1, 3$ and $i \in \mathbb{N}$. Then $\partial_{\pm} db_j^{(i)}$ is the intersection of $\partial DB_j^{(i)}$ with the one-skeleton of $\phi_i \circ p_T(\Delta_T^{\pm})$, where Δ_T^- is the image of the triangulation $\Delta_{T_0}^-$ defined below Corollary 2.3 under the inclusion $M_{T_0} \rightarrow M_T$, and Δ_T^+ is its reflected image.

By Lemma 2.4, $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ preserves triangulations, and the discussion above implies that the other gluing maps do as well. From Figure 5 it is apparent that the cusps of $F^{(0)}$ corresponding to p_1 and p_3 each contain only one end of an edge of the triangulation \mathcal{T} which it inherits from \mathcal{F} . Therefore $i_0(\partial a_1) = \partial_- db_1^{(1)} \cup \partial_- db_3^{(1)}$.

It then follows from (3) that

$$\alpha_1 = a_1 \cup db_1^{(1)} \cup \dots \cup db_1^{(n)} \cup \bar{a}_1 \cup db_3^{(n)} \cup \dots \cup db_3^{(1)}$$

is a closed geodesic on T_1 which meets β_1 once, at right angles. Therefore by Lemma 4.8, $\Re(m(T_1)) = 0$, and up to the action of $\mathrm{PGL}_2(\mathbb{Z})$, $m(T_1) = i(2 + 4n\sqrt{2})$.

A similar argument will give $m(T_2)$. Let \mathcal{A}_2 be the collection of arcs in A_2 which are the projections of internal edges of the squares comprising it. From Figure 8, one finds that \mathcal{A}_2 consists of five arcs evenly spaced around A_2 , each joining one component of ∂A_2 to the other and perpendicular to ∂A_2 at each endpoint. For $j = 2, 4$, we define a collection of arcs $DB_j \subset DB_j$ analogously, and take $DB_j^{(i)} = \phi_i \circ p_T(DB_j)$. Let $\partial_{\pm} DB_j^{(i)} = DB_j^{(i)} \cap \partial_{\pm} DB_j^{(i)}$, and note that the points of $\partial_{\pm} DB_j^{(i)}$ are the points of intersection of $\partial DB_j^{(i)}$ with the one-skeleton of $\phi_i \circ p_T(\Delta_T^{\pm})$.

For the same reasons as above, $i_+^{(0)}(i_-^{(0)})^{-1}$ takes ∂A_2 to $\partial_- DB_2^{(1)} \cup \partial_- DB_4^{(1)}$, and the other gluing maps take the $\partial_+ DB_j^{(i)}$ to $\partial_- DB_j^{(i+1)}$ for the appropriate i and j . Then the collection

$$\mathcal{A}_2 \cup DB_2^{(1)} \cup \dots \cup DB_2^{(n)} \cup \bar{\mathcal{A}}_2 \cup DB_4^{(n)} \cup \dots \cup DB_4^{(1)}$$

consists of a disjoint union of up to five closed geodesics, each of which meets β_2 perpendicularly in at most 5 points.

Fix a component α_2 of the collection above, let k be the intersection number of α_2 with β_2 , and let \tilde{T}_2 be the k -fold cover of T_2 dual to α_2 . Then β_2 lifts to \tilde{T}_2 , and any lift intersects the preimage $\tilde{\alpha}_2$ of α once, perpendicularly. Computing the modulus of \tilde{T}_2 using this pair, we obtain $\pm k \cdot i(2 + 4n\sqrt{2})/5$. This is $\mathrm{PGL}_2(\mathbb{Q})$ -equivalent to $i(2 + 4n\sqrt{2})/5$ and $m(T_2)$, as claimed. \square

Since commensurable hyperbolic manifolds have commensurable cusps, the collection of $\mathrm{PGL}_2(\mathbb{Q})$ -orbits of the cusp parameters of M is a commensurability invariant. This fact and the lemma below provide another means for verifying Proposition 4.5.

Lemma 4.16. *Suppose $z = i(m+n\sqrt{2})$ is $\mathrm{PGL}_2(\mathbb{Q})$ -equivalent to $z' = i(m+n'\sqrt{2})$, where $m, n, n' \in \mathbb{Q}$ and $m \neq 0$. Then $n' = \pm n$.*

Proof. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Q})$ takes z to z' . After clearing denominators (which does not change the action by Möbius transformations), we may assume that $a, b, c, d \in \mathbb{Z}$. We have

$$\frac{ai(m+n\sqrt{2})+b}{ci(m+n\sqrt{2})+d} = i(m+n'\sqrt{2}).$$

Multiplying by the denominator on the left, and collecting the real and imaginary parts, we find

$$m(a-d) + (an-dn')\sqrt{2} = 0 \quad b + c(m^2 + 2nn') + cm(n'+n)\sqrt{2} = 0$$

Since 1 and $\sqrt{2}$ are linearly independent over \mathbb{Q} , the left-hand equation above implies that $m(a-d) = 0$ and $an-dn' = 0$. Since $m \neq 0$, the first equation implies $a = d$. Then the second equation implies $n = n'$ unless $a = d = 0$. But in this case, $c \neq 0$ since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Q})$. Hence, using the coefficient of $\sqrt{2}$ in the right-hand equation above, we find $n' = -n$. \square

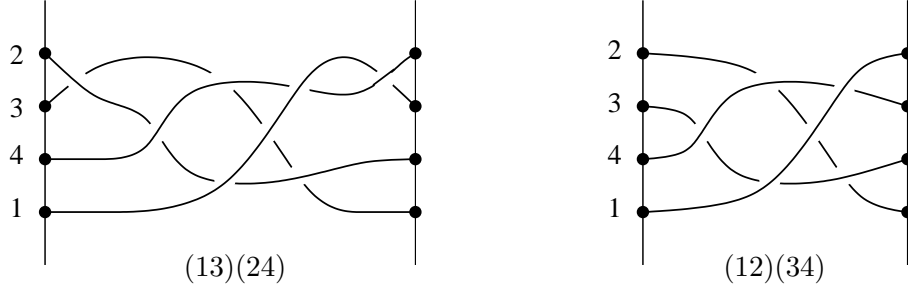


FIGURE 10. The mutations as braids.

5. COMMENSURABLE MUTANTS

In this section and the next, we will consider the relationship between L_n and links obtained from it by mutation along the separating spheres $S^{(i)}$, $0 \leq i \leq n$, from Definition 3.7(3). As remarked in the introduction, a *mutation* of a sphere with four marked points is a mapping class of order 2 which acts on the set of marked points by an even permutation. Such a map is uniquely determined up to isotopy by its image in the symmetric group on the marked points — this is Corollary ?? of Appendix ??.

Let S be the sphere obtained by compactifying each cusp of $F^{(0)} = \mathcal{H}/\Lambda$, defined in Lemma 2.4, with a single point. Label each additional point by a number between 1 and 4, according to the parabolic \mathfrak{p}_i corresponding to the cusp it compactifies. We will refer to the mutations of $(S, \{1, 2, 3, 4\})$ by their permutation representations in the alternating group on 4 letters.

Corollary 3.10 and the definition of $F^{(i)}$ together imply that for each i between 0 and n , $\phi_{c^{2i}} \circ f_n$ extends to a homeomorphism $S^{(i)} \rightarrow S$ taking marked points to marked points, which preserves the numbering of $S^{(i)} \cap L_n$ from Definition 3.7(3).

Lemma 5.1. *With L_n projected as in Figure 1, the link obtained from it by the mutation (13)(24) (respectively, (12)(34)) along $S^{(i)}$ is obtained by cutting L_n along $S^{(i)}$ and inserting the braid on the left (resp. right) of Figure 10.*

Proof. As we asserted in the Introduction, with L_n projected in the plane and $S^{(i)}$ represented by a vertical line as in Figure 1, the points of $S^{(i)} \cap L_n$ read off as 2, 3, 4, 1 from top to bottom. This follows from Definition 3.7(3) and the projections of S and T_0 pictured in Figure 2.

If we regard a horizontal axis as running midway between points 3 and 4 so that points 2 and 1 are also equidistant from it, then the 180-degree rotation in this axis restricts on $S^{(i)}$ to an involution acting on the marked points as the permutation (12)(34). There is a homeomorphism $R: S^{(i)} \times I \rightarrow S^{(i)} \times I$, which preserves slices $S^2 \times \{t\}$ and restricts on each to rotation by $-180 \cdot t$ degrees in the horizontal axis. This interpolates between the identity, on $S^2 \times \{0\}$, and the inverse of (12)(34) on $S^2 \times \{1\}$, although it does not preserve marked points for $0 < t < 1$.

Take $S^{(i)}$ to bound balls B^- , containing S , and B^+ , containing \bar{S} , in S^3 . We will denote by $L_-^{(i)}$ the tangle $L_n \cap B^-$, so that the link L' obtained by the mutation (12)(34) along $S^{(i)}$ is $L_-^{(i)} \cup_{(12)(34)} (L_n \cap B^+)$. Let C be a collar of $S^{(i)}$ in B^- , small enough that it intersects $L_-^{(i)}$ in the collection of horizontal arcs $\{\{j\} \times I \mid j \in$

$\{1, 2, 3, 4\}$. A homeomorphism $B^- \cup_{m_2} B^+ \rightarrow S^3$ may be defined as the identity on B^+ and the complement of C in B^- , and as R on C . By the definition of R , the image of $L_-^{(i)} \cap C$ under R is as pictured on the right-hand side of Figure 10, thus the image of L' in S^3 under the homeomorphism described above is as stated in the lemma.

The reader should note that the braid on the left-hand side of Figure 10 is the conjugate of the braid on the right-hand side by a left-handed half-twist exchanging the points 2 and 3. This reflects the fact that the conjugate of (12)(34), by any homeomorphism of $(S, \{1, 2, 3, 4\})$ which exchanges 2 and 3 and fixes 1 and 4, is a mapping class of order 2 acting on the marked points as (13)(24); hence such a conjugate is (13)(24). The conjugating braid in Figure 10 tracks the marked points under an isotopy $S \times I \rightarrow S$ taking the simplest such conjugator to the identity. The conclusion for (13)(24) thus follows as it did above for (12)(34). \square

After adjoining the braid realizing (13)(24) to the tangle S of Figure 2, there is an obvious isotopy removing two crossings between the arcs labeled e and v there. The diagram which results is symmetric under the rotation of B^3 given by 180-degree rotation about the horizontal axis described in the proof of Lemma 5.1. An attribute which we ascribed to (13)(24) below Theorem 2 thus follows:

Corollary 5.2. *The mutation (13)(24) extends over (B^3, S) .*

By [28, Theorem 2.2], each mutation of $(S, \{1, 2, 3, 4\})$ is realized by an isometry of $F^{(0)}$. (We say an isometry of $F^{(0)}$ *realizes* the mapping class of $(S, \{1, 2, 3, 4\})$ containing the homeomorphism to which it extends.) The lemma below identifies lifts to $\mathrm{PSL}_2(\mathbb{R})$ of the isometries realizing (13)(24) and (12)(34).

Lemma 5.3. *Define*

$$\mathfrak{m}_1 = \begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix} \qquad \mathfrak{m}_2 = \begin{pmatrix} 0 & \sqrt{5} \\ \frac{-1}{\sqrt{5}} & 0 \end{pmatrix}$$

Each of \mathfrak{m}_1 and \mathfrak{m}_2 normalizes Λ , and the induced isometries $\phi_{\mathfrak{m}_1}$ and $\phi_{\mathfrak{m}_2}$ of $F^{(0)}$ realize (13)(24) and (12)(34), respectively.

Proof. Since each of \mathfrak{m}_1 and \mathfrak{m}_2 has trace equal to zero, it has order 2 in $\mathrm{PSL}_2(\mathbb{C})$. Their actions by conjugation described below, on the generators \mathfrak{p}_1 , \mathfrak{p}_2 , and \mathfrak{p}_3 for Λ defined above Lemma 2.4, may be verified by direct computation.

$$\begin{aligned} \mathfrak{p}_1^{\mathfrak{m}_1} &= \mathfrak{p}_3^{-1} & \mathfrak{p}_2^{\mathfrak{m}_1} &= \mathfrak{p}_4^{-1} \\ \mathfrak{p}_1^{\mathfrak{m}_2} &= \mathfrak{p}_2 & \mathfrak{p}_3^{\mathfrak{m}_2} &= \mathfrak{p}_4^{\mathfrak{p}_1^{-1}} \end{aligned}$$

Here $\mathfrak{p}_4 = \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3^{-1}$ is the parabolic described in Remark 1 below Lemma 2.4. It follows that \mathfrak{m}_1 and \mathfrak{m}_2 normalize Λ and induce isometries $\phi_{\mathfrak{m}_1}$ and $\phi_{\mathfrak{m}_2}$, respectively, of $F^{(0)} = C(\Lambda)$.

Each of $\phi_{\mathfrak{m}_1}$ and $\phi_{\mathfrak{m}_2}$ has order 2, since \mathfrak{m}_1 and \mathfrak{m}_2 have order 2, and their extensions to S act on the set of marked points as described in the statement of the lemma. Its conclusion therefore follows from Corollary ?? \square

Corollary 5.4. *For $j = 1, 2$ and $i \in \mathbb{Z}$, let $\mathfrak{m}_j^{(i)} = \mathfrak{c}^{-2i} \mathfrak{m}_j \mathfrak{c}^{2i}$. Each of $\mathfrak{m}_1^{(i)}$ and $\mathfrak{m}_2^{(i)}$ normalizes $\Lambda^{(i)}$ from Definition 3.8(1), and the induced isometries of $F^{(i)}$ realize (13)(24) and (12)(34), respectively, under the restriction of f_n described in Corollary 3.10.*

Lemma 5.1 gives a prescription for describing the links obtained from L_n by the mutations (13)(24) and (12)(34). The Proposition below is the analog of Proposition 3.9 for the mutants, describing hyperbolic manifolds to which their complements are homeomorphic.

Proposition 5.5. *Suppose L' is obtained from L_n by the mutation (13)(24) along each member of a subcollection of the $S^{(i)}$. Let $M' = C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \dots \cup C(\Gamma_T^{(n)}) \cup C(\overline{\Gamma}_S)$, where for each i such that $S^{(i)}$ is not a mutation sphere the gluing map is as in Proposition 3.9, and otherwise is given by*

$$\begin{aligned} & \iota_+^{(i)} \phi_{m_1^{(i)}}(\iota_-^{(i)})^{-1} \text{ for } 0 \leq i < n, \text{ or} \\ & \phi_r \iota_-^{(0)} \phi_{c^{2n}} \phi_{m_1^{(n)}}(\iota_-^{(n)})^{-1} \text{ otherwise.} \end{aligned}$$

There is a homeomorphism $h' : S^3 - L' \rightarrow M'$ whose restriction to each complementary component of the collection $\{S^{(i)}\}$ agrees with that of h_n .

The analogous assertion holds for L' obtained from L_n by the mutation (12)(34) along each member of a subcollection of the $S^{(i)}$, replacing $\phi_{m_1^{(i)}}$ above by $\phi_{m_2^{(i)}}$.

This follows immediately from Proposition 3.9 and Lemma 5.4. The lemma below describes the effect at the level of Kleinian groups, of cutting a hyperbolic manifold along a separating totally geodesic surface and regluing by an isometry.

Lemma 5.6. *Suppose Γ_0 and Γ_1 meet cute along a plane \mathcal{K} , and let Θ , $E = \mathcal{K}/\Theta$, and ι_0 and ι_1 be as in Lemma 3.2. If n normalizes Θ , then $\langle \Gamma_0, \Gamma_1^n \rangle$ is a Kleinian group, and there is an isometry*

$$C(\Gamma_0) \cup_{\iota_1 \phi_n \iota_0^{-1}} C(\Gamma_1) \rightarrow C(\langle \Gamma_0, \Gamma_1^n \rangle)$$

which restricts on $C(\Gamma_0)$ to the natural map, and on $C(\Gamma_1)$ to $\phi_n : C(\Gamma_1) \rightarrow C(\Gamma_1^n)$ followed by the natural map.

Proof. Since n normalizes Θ , it preserves \mathcal{K} ; hence Γ_0 and Γ_1^n meet cute along \mathcal{K} , and Lemma 3.2 applies. Thus $\langle \Gamma_0, \Gamma_1^n \rangle$ is a Kleinian group, and in particular, the natural maps $C(\Gamma_0) \rightarrow C(\langle \Gamma_0, \Gamma_1^n \rangle)$ and $C(\Gamma_1^n) \rightarrow C(\langle \Gamma_0, \Gamma_1^n \rangle)$ determine an isometry

$$C(\Gamma_0) \cup_{n\iota_1\iota_0^{-1}} C(\Gamma_1^n) \rightarrow C(\langle \Gamma_0, \Gamma_1^n \rangle).$$

Here we are using $n\iota_1 : E \rightarrow C(\Gamma_1^n)$ to refer to the natural map. It is now an exercise in definition-chasing to show that $n\iota_1 \circ \phi_n = \phi_n \circ \iota_1$, whence the map

$$C(\Gamma_0) \cup_{\iota_1 \phi_n \iota_0^{-1}} C(\Gamma_1) \rightarrow C(\Gamma_0) \cup_{n\iota_1\iota_0^{-1}} C(\Gamma_1^n),$$

defined as the identity on $C(\Gamma_0)$ and ϕ_n on $C(\Gamma_1)$, is well-defined. The lemma follows. \square

The main goal of this section is to prove that Γ_n is commensurable with the Kleinian groups associated to each of its mutants by (13)(24). There is a group that contains them all, generated by reflections in a certain family of hyperplanes.

Below, let \mathcal{B}_0 be the open half-ball in \mathbb{H}^3 bounded by the Euclidean hemisphere of unit radius centered at $0 \in \mathbb{C}$, and let $\mathcal{B}_j = c^{-j}(\mathcal{B}_0)$, where c is as defined in Lemma 3.3. Recall that we have defined \mathcal{H} as the geodesic hyperplane of \mathbb{H}^3 with ideal boundary $\mathbb{R} \cup \{\infty\}$. If w and z are complex numbers, we will take $w\mathcal{H} + z$ to be the hyperplane with ideal boundary $(w\mathbb{R} + z) \cup \{\infty\}$.

- Definition 5.7.** (1) Let \mathbf{f}_0 be obtained by first reflecting in $i\mathcal{H}$ and then in $i\mathcal{H} + 1/2$.
(2) Let \mathbf{b}_0 be obtained by first reflecting in $\mathcal{H} + i/2$ and then in $\partial\mathcal{B}_0$.
(3) For $i \geq 0$, let \mathbf{a}_i be obtained by reflecting in $i\mathcal{H} + 1/2$ and then in $\partial\mathcal{B}_i$.

Since $i\mathcal{H}$ and $i\mathcal{H} + 1/2$ are parallel and share the ideal point ∞ , f_0 is a parabolic isometry fixing ∞ . $\mathcal{H} + i/2$ meets $\partial\mathcal{B}_0$ at an angle of $\pi/3$, so \mathbf{b}_0 is an elliptic isometry of order 3 rotating around the geodesic of intersection. For the same reason, \mathbf{a}_i is an elliptic element of order 3 rotating around the geodesic $i\mathcal{H} + 1/2 \cap \partial\mathcal{B}_i$ for each $i \geq 0$.

Lemma 5.8. *Let G_n be the group generated by reflections in the face of P_n , where*

$$P_n = \left\{ (z, t) \in \mathbb{H}^3 \mid 0 \leq \Re(z) \leq 1/2, -n\sqrt{2} \leq \Im(z) \leq 1/2 \right\} - \left(\bigcup_{k=0}^n \mathcal{B}_k \right).$$

Then G_n contains \mathbf{a}_i for $0 \leq i \leq 2n$, as well as \mathbf{f}_0 and \mathbf{b}_0 .

Proof. By its definition, P_n is cut out by $\mathcal{H} + i/2$, $i\mathcal{H}$, $i\mathcal{H} + 1/2$, $\mathcal{H} - n \cdot i\sqrt{2}$, and the $\partial\mathcal{B}_k$, $0 \leq k \leq n$. It therefore follows from the definitions above that G_n contains \mathbf{f}_0 , \mathbf{b}_0 , and the \mathbf{a}_i for $0 \leq i \leq n$. It remains to establish that G_n contains \mathbf{a}_i for $n < i \leq 2n$.

Note that the bounding hyperplane $\mathcal{H} - n \cdot i\sqrt{2}$ for P_n is the image of \mathcal{H} under c^{-n} , so that reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ is given by $c^{-n}rc^n$, where r is the reflection through \mathcal{H} . Using the property of r observed above Lemma 3.3, the conjugate of any element $x \in \mathrm{PSL}_2(\mathbb{C})$ by the reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ is:

$$(6) \quad c^{-n}rc^nxc^{-n}rc^n = c^{-2n}\bar{x}c^{2n}$$

We further observe that c conjugates \mathbf{a}_i to \mathbf{a}_{i-1} for $i \geq 1$, since $c(i\mathcal{H} + 1/2) = i\mathcal{H} + 1/2$ and $c(\mathcal{B}_i) = \mathcal{B}_{i-1}$, and we note that $\bar{\mathbf{a}}_0 = \mathbf{a}_0$. Thus:

$$(7) \quad \overline{c^i\mathbf{a}_i c^{-i}} = \bar{\mathbf{a}}_0 = \mathbf{a}_0 = c^i\mathbf{a}_i c^{-i} \Rightarrow c^{-2i}\bar{\mathbf{a}}_i c^{2i} = \mathbf{a}_i$$

For $0 \leq i \leq n$, it follows that the conjugate of \mathbf{a}_i by reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ is:

$$c^{-2n}\bar{\mathbf{a}}_i c^{2n} = c^{-2(n-i)}\mathbf{a}_i c^{2(n-i)} = \mathbf{a}_{2n-i} \in G_n.$$

Therefore G_n contains \mathbf{a}_i for $n \leq i \leq 2n$ as well, and the lemma is proved. \square

Since \mathcal{H} meets both $\partial\mathcal{B}_0$ and $i\mathcal{H} + 1/2$ at right angles, it does the same for the fixed geodesic of \mathbf{a}_0 and is therefore preserved by \mathbf{a}_0 . In fact, the following description of $\mathbf{a}_0 \in \mathrm{PSL}_2(\mathbb{C})$ is easily obtained from its definition:

$$\mathbf{a}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

In particular, \mathbf{a}_0 acts on the ideal points of $\mathcal{P}_1 \cap \mathcal{P}_2$ by $0 \mapsto 1 \mapsto \infty \mapsto 0$. Similarly, it is easy to see that $f_0(z, t) = (z + 1, t)$

Then the face pairings \mathbf{f} (defined in Lemma 2.3) and \mathbf{s} (defined in Lemma 2.2), which are equal, are obtained from \mathbf{f}_0 by conjugating by \mathbf{a}_0 :

$$(8) \quad \mathbf{s} = \mathbf{f} = \mathbf{a}_0 \mathbf{f}_0 \mathbf{a}_0^{-1}.$$

One may use similar analyses to establish the following.

$$(9) \quad \mathbf{t} = (\mathbf{b}_0 \mathbf{a}_0)^{-1} \mathbf{f}_0 (\mathbf{b}_0 \mathbf{a}_0) \mathbf{a}_0 \quad \mathbf{g} = (\mathbf{a}_0^{-1} \mathbf{a}_1) \mathbf{f}_0^{-1} (\mathbf{a}_0^{-1} \mathbf{a}_1)^{-1} \quad \mathbf{h} = \mathbf{a}_1 \mathbf{a}_0 \mathbf{f}_0^{-1} \mathbf{a}_1$$

The main group-theoretic fact of this section extends these observations.

Proposition 5.9. *For each $n \in \mathbb{N}$, G_n contains Γ_n and $\mathfrak{m}_1^{(i)}$ for $0 \leq i \leq n$.*

Proof. We recall from Proposition 3.11 that $\Gamma_n = \langle \Gamma_S, \Gamma_T^{(1)}, \dots, \Gamma_T^{(n)}, \overline{\Gamma_S}^{c^{-2n}} \rangle$, where by Definition 3.8(2), $\Gamma_T^{(i)} \doteq \Gamma_T^{c^{-2(i-1)}}$ for each i between 1 and n . Furthermore, by Lemma 3.3, $\Gamma_T = \langle \Gamma_{T_0}, \overline{\Gamma_{T_0}}^{c^{-2}} \rangle$.

It is a direct consequence of the descriptions (8) and (9) above that $\Gamma_S < G_n$ and $\Gamma_{T_0} < G_n$. Furthermore, since \mathfrak{f}_0 commutes with \mathfrak{c} and $\bar{\mathfrak{f}}_0 = \mathfrak{f}_0$, (7) implies for instance that

$$\mathfrak{c}^{-2}\bar{\mathfrak{f}}\mathfrak{c}^2 = \mathfrak{c}^{-2}(\bar{\mathfrak{a}}_0\bar{\mathfrak{f}}_0\bar{\mathfrak{a}}_0^{-1})\mathfrak{c}^2 = \mathfrak{a}_2\mathfrak{f}_0\mathfrak{a}_2^{-1} \in \Gamma_n,$$

since $\bar{\mathfrak{a}}_0 = \mathfrak{a}_0$ and $\mathfrak{c}^{-2}\mathfrak{a}_0\mathfrak{c}^2 = \mathfrak{a}_2$. Using the same strategy, we find:

$$\mathfrak{c}^{-2}\bar{\mathfrak{g}}\mathfrak{c}^2 = (\mathfrak{a}_2^{-1}\mathfrak{a}_1)\mathfrak{f}_0^{-1}(\mathfrak{a}_2^{-1}\mathfrak{a}_1)^{-1} \in G_n \text{ and } \mathfrak{c}^{-2}\bar{\mathfrak{h}}\mathfrak{c}^2 = \mathfrak{a}_1\mathfrak{a}_2\mathfrak{f}_0^{-1}\mathfrak{a}_1 \in G_n$$

Thus G_n contains $\Gamma_T = \Gamma_T^{(1)}$. Since conjugation by \mathfrak{c}^{-1} takes \mathfrak{a}_i to \mathfrak{a}_{i+1} , and $\mathfrak{a}_i \in \Gamma_n$ for each i between 0 and $2n$, it follows from the descriptions above and in (8) and (9) that each $\Gamma_T^{(i)}$, $1 \leq i \leq n$. Finally the relation (6) immediately implies that $\overline{\Gamma_S}^{c^{-2n}} < G_n$, and we have established that $\Gamma_n < G_n$.

To show that G_n contains the elements $\mathfrak{m}_1^{(j)}$ for each j between 0 and n , we observe that the element obtained by reflecting first in $\partial\mathcal{B}_0$ and then in $i\mathcal{H}$ is the rotation of order 2 described by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This is well known to generate $\mathrm{PSL}_2(\mathbb{Z})$, along with \mathfrak{a}_0 . Since $\mathfrak{m}_1 \in \mathrm{PSL}_2(\mathbb{Z})$, it follows that $\mathfrak{m}_1 \in G$.

We note that \mathfrak{c}^{-2j} preserves $i\mathcal{H}$ and takes \mathcal{B}_0 to \mathcal{B}_{2j} , and that \mathcal{B}_{2j} intersects P_n , for $j \leq n/2$, and intersects its image under reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ for $n/2 \leq j \leq n$. Thus for each j between 0 and n , the rotation obtained by reflecting first in $\partial\mathcal{B}_{2j}$ and then in $i\mathcal{H}$ is contained in G_n . If \mathfrak{m}_1 is expressed as a word in the two elements described in the paragraph above, then $\mathfrak{c}^{-2j}\mathfrak{m}_1\mathfrak{c}^{2j}$ is expressed as the same word in \mathfrak{a}_{2j} and the rotation obtained from $\partial\mathcal{B}_{2j}$ as above. The lemma follows. \square

Theorem 2 follows from Lemma 5.6 and Proposition 5.9.

Proof of Theorem 2. Suppose first that L' is obtained from L_n by mutation along a single sphere $S^{(i)}$, $1 \leq i \leq n-1$. (We may assume that $i \neq 0$ or n , since Corollary 5.2 implies that mutating by (13)(24) along $S^{(0)}$ or $S^{(n)}$ recovers L_n .) We will regard Γ_n as $\langle \Gamma_-^{(i)}, \Gamma_+^{(i)} \rangle$, where $\Gamma_{\pm}^{(i)}$ is as defined in Corollary 3.12. Then $\Gamma_-^{(i)}$ and $\Gamma_+^{(i)}$ meet cute along $\mathfrak{c}^{(-2i)}(\mathcal{H})$, and

$$\mathrm{Stab}_{\Gamma_-^{(i)}}(\mathfrak{c}^{-2i}(\mathcal{H})) = \mathrm{Stab}_{\Gamma_+^{(i)}}(\mathfrak{c}^{-2i}(\mathcal{H})) = \Lambda^{(i)},$$

from Definition 3.8(1). By Corollary 5.4, $\mathfrak{m}_1^{(i)}$ normalizes $\Lambda^{(i)}$, therefore Lemma 5.6 describes an isometry

$$C(\Gamma_-^{(i)}) \cup_{\phi_{\mathfrak{m}_1^{(i)}}} C(\Gamma_+^{(i)}) \rightarrow C(\langle \Gamma_-^{(i)}, (\Gamma_+^{(i)})^{\mathfrak{m}_1^{(i)}} \rangle).$$

By Corollary 3.12 and Proposition 5.5, the object on the left above is isometric to the hyperbolic manifold M' homeomorphic to $S^3 - L'$. Since, by Proposition 5.9, $\langle \Gamma_-^{(i)}, (\Gamma_+^{(i)})^{\mathfrak{m}_1^{(i)}} \rangle < G_n$, it follows that M' is commensurable with M_n .

The general case now follows by induction. Given a link L' obtained from L by the mutation (13)(24) along spheres $S^{(i_0)}, S^{(i_1)}, \dots, S^{(i_k)}$, where $0 \leq i_0 < i_1 < \dots < i_k \leq n$, let L'' be the link obtained from L by mutating along all the spheres above but $S^{(i_0)}$. We inductively assume the following: there is a group $\Gamma'' < G_n$

such that $\Gamma_-^{(i_1)}$ and Γ'' meet cute along $c^{-2i_1}(\mathcal{H})$, and there is an isometry from the hyperbolic manifold M'' supplied by Proposition 5.5, which is homeomorphic to $S^3 - L''$, to $C(\langle \Gamma_-^{(i_1)}, \Gamma'' \rangle)$ which restricts on $C(\Gamma_-^{(i_1)})$ to the natural map. Take

$$\Gamma_+ = \langle \Gamma_T^{(i_0+1)}, \dots, \Gamma_T^{(i_1)}, \Gamma'' \rangle,$$

and let $\Gamma' = \Gamma_+^{m_1^{(i_0)}} < G_n$. As above, Γ' and $\Gamma_-^{(i_0)}$ meet cute along $c^{-2i_0}(\mathcal{H})$, and Lemma 5.6 describes an isometry from M' to $C(\langle \Gamma_-^{(i_0)}, \Gamma' \rangle)$ which restricts on $C(\Gamma_-^{(i_0)})$ to the natural map and on its complement to $\phi_{m_1^{(i_0)}}$ followed by the natural map. The theorem follows. \square

6. INCOMMENSURABLE MUTANTS

The description of Lemma 5.3 might lead one to suspect that the mutations (13)(24) and (12)(34) of $F^{(0)}$ act very differently at the level of Kleinian groups. Indeed, it will follow from Proposition 6.3 below, together with Proposition 4.2, that $S^3 - L_n$ is incommensurable with the complement of any link obtained from it by the mutation (12)(34) along a subcollection of the $S^{(i)}$. In fact, we consider it likely that no two such mutants are commensurable unless they are isometric.

We lack the tools to prove the assertion above — mutants are notoriously difficult to distinguish because they share so many invariants. Examples are the invariant trace field (which in this case is the trace field $\mathbb{Q}(i, \sqrt{2})$) and (probably) the Bloch invariant (we will prove this particular case in Proposition 6.4 below), as remarked in 1 below the statement of Theorem 3. However, in this section we will describe families of mutants whose members have different cusp parameters and are therefore mutually incommensurable.

Definition 6.1. For $I = (a_0, a_1, \dots, a_n) \in \{0, 1\}^{n+1}$, let L_I be the link obtained from L_n by mutation along $S^{(i)}$ for each i such that $a_i = 1$, in each case using the mutation (12)(34). (Note that by this definition, $L_{(0, \dots, 0)} = L_n$.) Let M_I be the hyperbolic manifold supplied by Proposition 5.5, homeomorphic to $S^3 - L_I$.

The proposition below follows from Corollary 5.4 and Lemma 5.6 in a similar way to Proposition 5.9. The statement recalls Propositions 3.9 and 3.11, and Corollary 3.12.

Proposition 6.2. For $I = (a_0, a_1, \dots, a_n) \in \{0, 1\}^{n+1} - \{(0, \dots, 0)\}$, let i_0 be the smallest index such that $a_{i_0} = 1$. If $i_0 = n$, define $\Gamma_I = \langle \Gamma_-^{(n)}, (\overline{\Gamma}_S^{c^{-2n}})^{m_2^{(n)}} \rangle$. There is an isometry $M_I \rightarrow C(\Gamma_I)$ which restricts on $C(\Gamma_-^{(n)})$ to the natural map and on $C(\overline{\Gamma}_S)$ to $\phi_{m_2^{(n)}} \circ \phi_{c^{-2n}}$, followed by the natural map.

If $i_0 < n$, there is a Kleinian group Γ_+ so that $(\Gamma_T^{(i_0+1)})^{m_2^{(i_0)}}$ and Γ_+ meet cute along $m_2^{(i_0)} c^{-2(i_0+1)}(\mathcal{H})$, and so that defining $\Gamma_I = \langle \Gamma_-^{(i_0)}, (\Gamma_T^{(i_0+1)})^{m_2^{(i_0)}}, \Gamma_+ \rangle$, there is an isometry $M_I \rightarrow C(\Gamma_I)$ which restricts on $C(\Gamma_-^{(i_0)})$ to the natural map and on $C(\Gamma_T^{(i_0)})$ to $\phi_{m_2^{(i_0)}}$, followed by the natural map.

Proof. The case $i_0 = n$ follows as in the proof of Proposition 3.11. Recall that we showed there that $\Gamma_-^{(n)}$ and $\overline{\Gamma}_S^{c^{-2n}}$ meet cute along $c^{-2n}(\mathcal{H})$, and applied Lemma 3.2 to produce the desired isometry. By Lemma 5.4, $m_2^{(n)}$ normalizes $\Lambda^{(n)}$; hence Lemma 5.6 implies the conclusion in this case.

We prove the case $i_0 < n$ by induction on the number of indices i such that $a_i = 1$. If i_0 is the unique such index, let $\Gamma_-^{(i_0)}$ and $\Gamma_+^{(i_0)}$ be as described in Corollary 3.12. By Corollary 3.12, the isometry $M_n \rightarrow C(\Gamma_n)$ restricts on each of $C(\Gamma_-^{(i_0)})$ and $C(\Gamma_+^{(i_0)})$ to the natural map. Then by Lemma 5.6, the map which restricts on $C(\Gamma_-^{(i_0)})$ to the natural map, and on $C(\Gamma_+^{(i_0)})$ to $\phi_{\mathfrak{m}_2^{(i_0)}}$ followed by the natural map, determines an isometry

$$M_I \rightarrow C(\langle \Gamma_-^{(i_0)}, (\Gamma_+^{(i_0)})_{\mathfrak{m}_2^{(i_0)}} \rangle).$$

We note that $(\Gamma_+^{(i_0)})_{\mathfrak{m}_2^{(i_0)}} = \langle (\Gamma_T^{(i_0)})_{\mathfrak{m}_2^{(i_0)}}, (\Gamma_+^{(i_0+1)})_{\mathfrak{m}_2^{(i_0)}} \rangle$, and that the isometry above restricts on $C(\Gamma_T^{(i_0)})$ to $\phi_{\mathfrak{m}_2^{(i_0)}}$ followed by the natural map. Thus after defining $\Gamma_+ = (\Gamma_+^{(i_0+1)})_{\mathfrak{m}_2^{(i_0)}}$, the conclusion of the proposition is satisfied.

If there exists $i_1 > i_0$ such that $a_{i_1} = 1$, we assume by induction that the conclusion of the proposition applies to the ordered tuple I' which has all of the same entries as I except for the i_0 entry, which is 0. Let Γ'_+ be the Kleinian group supplied by applying the proposition to $M_{I'}$, define

$$\Gamma_+^0 = \langle \Gamma_T^{(i_0+2)}, \dots, \Gamma_T^{(i_1)}, (\Gamma_T^{(i_1+1)})_{\mathfrak{m}_2^{(i_1)}}, \Gamma'_+ \rangle,$$

and let $\Gamma_+ = (\Gamma_+^0)_{\mathfrak{m}_2^{(i_0)}}$. (If $i_1 = i_0 + 1$, then we define Γ_+^0 to be generated by only the last two subgroups above.)

As above, $\Gamma_-^{(i_0)}$ and $\Gamma_T^{(i_0+1)}$ meet cute along $\mathfrak{c}^{-2i_0}(\mathcal{H})$, and by induction, $\Gamma_T^{(i_0+1)}$ and Γ_+^0 meet cute along $\mathfrak{c}^{-2(i_0+1)}(\mathcal{H})$. Then $\Gamma_-^{(i_0)}$ and $(\Gamma_T^{(i_0+1)})_{\mathfrak{m}_2^{(i_0)}}$ meet cute along $\mathfrak{c}^{-2i_0}(\mathcal{H})$, and $(\Gamma_T^{(i_0+1)})_{\mathfrak{m}_2^{(i_0)}}$ and Γ_+ meet cute along $\mathfrak{m}_2^{(i_0)}\mathfrak{c}^{-2(i_0+1)}(\mathcal{H})$. Hence Lemma 5.6 supplies an isometry

$$C(\Gamma_-^{(i_0)}) \cup_{\phi_{\mathfrak{m}_2^{(i_0)}}} C(\langle \Gamma_T^{(i_0+1)}, \Gamma_+^0 \rangle) \rightarrow C(\langle \Gamma_-^{(i_0)}, (\Gamma_T^{(i_0+1)})_{\mathfrak{m}_2^{(i_0)}}, \Gamma_+ \rangle)$$

given by the natural map on $C(\Gamma_-^{(i_0)})$ and $\phi_{\mathfrak{m}_2^{(i_0)}}$ followed by the natural map on the other side of $F^{(i_0)} = \mathfrak{c}^{-2i_0}(\mathcal{H})/\Lambda^{(i_0)}$. Since M_I is obtained from $M_{I'}$ by cutting along $F^{(i_0)}$ and regluing by $\phi_{\mathfrak{m}_2^{(i_0)}}$, the proposition holds by induction. \square

Proposition 6.3. *For fixed n and any $I \in \{0, 1\}^{n+1}$ with $I \neq (0, 0, \dots, 0)$, Γ_I has a nonintegral trace.*

Proof. Let $I = (a_0, \dots, a_n) \in \{0, 1\}^{n+1} - \{(0, \dots, 0)\}$, and let i_0 be the minimal index such that $a_{i_0} = 1$. If $i_0 = 1$ or n , Proposition 6.2 implies that Γ_I contains an element with the same trace as the matrix below, or its complex conjugate.

$$\mathfrak{m}_2 \mathfrak{t} \mathfrak{m}_2^{-1} \mathfrak{g} = \begin{pmatrix} -1 + \sqrt{2} + 11i + i\sqrt{2} & 1 - 7\sqrt{2} - 16i - 2i\sqrt{2} \\ \frac{1}{5}(2 - \sqrt{2} - 21i - 2i\sqrt{2}) & \frac{1}{5}(-2 + 12\sqrt{2} + 31i + 4i\sqrt{2}) \end{pmatrix}$$

The trace of $\mathfrak{m}_2 \mathfrak{t} \mathfrak{m}_2^{-1} \mathfrak{g}$ is not an algebraic integer, since the ring of integers of $\mathbb{Q}(i, \sqrt{2})$ is $\mathbb{Z}[i, \sqrt{2}]$. In the case $i_0 = 1$, the conjugate of this element by \mathfrak{m}_2 , $\mathfrak{t} \mathfrak{m}_2^{(0)} \mathfrak{g} (\mathfrak{m}_2^{(0)})^{-1}$, is contained in Γ_I , since $\mathfrak{m}_2^{(0)} = \mathfrak{m}_2$ has order 2. In the case $i_0 = n$, the conjugate of $\mathfrak{m}_2 \mathfrak{t} \mathfrak{m}_2^{-1} \mathfrak{g} = \mathfrak{m}_2 \bar{\mathfrak{t}} \mathfrak{m}_2^{-1} \bar{\mathfrak{g}}$ by \mathfrak{c}^{-2n} is contained in Γ_I .

In all other cases, Proposition 6.2 implies that Γ_I contains an element with the same trace as the matrix below.

$$\begin{aligned} \bar{h}(m_2 h m_2^{-1}) &= \begin{pmatrix} -2i\sqrt{2} & -3 + i\sqrt{2} \\ -3 - i\sqrt{2} & 3i\sqrt{2} \end{pmatrix} \begin{pmatrix} -3i\sqrt{2} & 15 - 5i\sqrt{2} \\ \frac{3+i\sqrt{2}}{5} & 2i\sqrt{2} \end{pmatrix}, \\ &= \begin{pmatrix} -71/5 & -20 - 30i\sqrt{2} \\ \frac{18}{5}(-2 + 3i\sqrt{2}) & 55 \end{pmatrix} \end{aligned}$$

The trace of this matrix is evidently not an algebraic integer. \square

For fixed n and any $I \in \{0, 1\}^{n+1}$, since M_n and M_I decompose along totally geodesic surfaces into isometric pieces, they have the same volume. (In fact, [28, Theorem 1.3] asserts that hyperbolic volume is always invariant under mutation.) It would follow from the classical ‘‘Dehn invariant sufficiency’’ conjecture that any two hyperbolic manifolds with the same volume are scissors congruent (again see [23], for instance). In our situation we will verify this explicitly.

Proposition 6.4. *For fixed n and any $I \in \{0, 1\}^{n+1}$, M_n and M_I have the same Bloch invariant.*

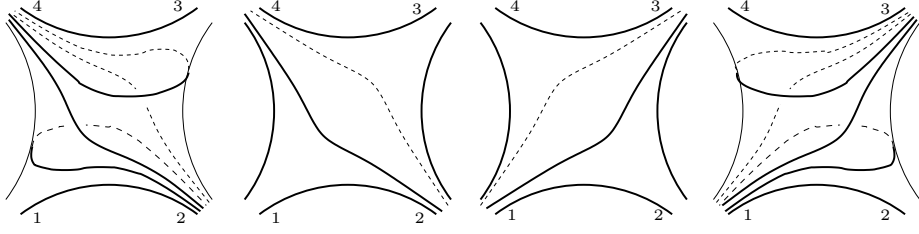
Proof. Recall from Lemma 2.4 that $F^{(0)}$ inherits a triangulation Δ_F from the fundamental domain \mathcal{F} for the action of Λ on \mathbf{H} pictured in Figure 5. From the figure, one finds that Δ_F has six edges, each a geodesic arc joining cusps of F . For example, the geodesic joining 0 and ∞ projects to an edge which joins cusp 1 to cusp 2. Of the other five edges, one joins 3 to 4, two join 2 to 4, and for each of 2 and 4 there is an edge joining it to itself.

Since ϕ_{m_2} exchanges 1 with 2 and 3 with 4, it does not preserve Δ_F . For instance, if e is the edge joining 2 to itself then $\phi_{m_2}(e)$ joins 1 to itself. Therefore for $I \in \{0, 1\}^{n+1} - \{(0, \dots, 0)\}$ the decomposition of M_I into octahedra and cuboctahedra is not an ‘‘ideal polyhedral decomposition’’. This is because when an entry $a_i = 1$, the copies of $C(\Gamma_S)$ or $C(\Gamma_T)$ on either side of $F^{(i)}$ are glued by $\phi_{m_2^{(i)}}$, which is conjugate to ϕ_{m_2} by $\phi^{(i)}$ and hence does not preserve Δ_F . Since Δ_F is the triangulation inherited from the external faces of the octahedron and cuboctahedron (by Lemma 2.4 again), it follows that external faces of the octahedra or cuboctahedra on either side of $F^{(i)}$ do not match up with each other.

It is possible to rectify this by gluing ‘‘flat’’ tetrahedra between copies of $C(\Gamma_S)$ and/or $C(\Gamma_T)$ joined by mutation. If \mathcal{T} is a flat tetrahedron glued to, say, $C(\Gamma_S)$ along two adjacent triangles in $\partial C(\Gamma_S)$, then $C(\Gamma_S) \cup \mathcal{T}$ is homeomorphic to $C(\Gamma_S)$, but in the induced triangulation of the boundary, the edge separating the triangles along which \mathcal{T} is glued has been replaced by an edge joining their two opposite vertices. For a more thorough exposition, see [24, §4].

Figure 11 illustrates a process by which Δ_F may be changed to its image under ϕ_{m_2} by a sequence of moves on edges. The edges of Δ_F are pictured on the left in bold. Moving left-to-right, at each stage two edges are replaced by edges transverse to them and disjoint from the remaining edges. After three such moves, the original triangulation has been changed to its image under ϕ_{m_2} .

Now suppose $I = (a_1, \dots, a_n) \in \{0, 1\}^{n+1} - \{0, \dots, 0\}$. For each $i < n$ such that $a_i = 1$, replace $C(\Gamma_T^{(i+1)})$ by its union with 6 flat tetrahedra, glued successively along $\partial_- C(\Gamma_T^{(i+1)})$ to realize the change of triangulations illustrated in Figure 11. The result is homeomorphic to $C(\Gamma_T^{(i+1)})$, since adding a flat tetrahedron does not

FIGURE 11. Interpolating between Δ_T , on the left, and $\phi_{m_2}(\Delta_T)$.

change the homeomorphism type, but the gluing induced by $\phi_{m_2^{i+1}}$ now preserves the triangulation.

It follows from the above that the Bloch invariant $\beta(M_I)$ may be calculated using the resulting polyhedral decomposition. This differs from the original by the addition of the cross ratio parameters of the flat tetrahedra. Each of these is equal to 2, since the triangulation of F is a projection of the Farey tessellation of \mathbf{H} . But in the Bloch group, $2 \cdot [2] = 0$ is a consequence of the relation $[z] = \left[\frac{z}{z-1} \right]$. Since the number of flat tetrahedra is a multiple of 6, the sum of their cross ratio parameters contributes nothing to the Bloch invariant. \square

Proposition 6.5. For $I = (a_0, a_1, \dots, a_n) \in \{0, 1\}^{n+1}$ and $i \in \{0, 1, \dots, n\}$, define

$$c_i = \sum_{j=0}^i a_j \pmod{2}.$$

Let T_1 be a cross section of the cusp of M_I such that $T_1 \cap M_S = A_1$ (as defined in Lemma 4.10), and let T_2 be a cross section of the cusp of M_I with $T_2 \cap M_S = A_2$. Up to the action of $\mathrm{PGL}_2(\mathbb{Q})$, their complex moduli are as follows.

$$m(T_1) = i \left[1 + 2 \sum_{i=1}^n \frac{2\sqrt{2}}{5^{c_{i-1}}} + \frac{1}{5^{c_n}} \right]$$

$$m(T_2) = i \left[\frac{1}{5} + 2 \sum_{i=1}^n \frac{2\sqrt{2}}{5^{(1-c_{i-1})}} + \frac{1}{5^{(1-c_n)}} \right]$$

Proof. We recall the decomposition of M_I , along the surfaces $F^{(i)}$, into a union of isometric copies of $C(\Gamma_S)$ and $C(\Gamma_T)$ as described in Proposition 5.5:

$$C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \dots \cup C(\Gamma_T^{(n)}) \cup C(\overline{\Gamma_S}^{c^{-2n}}) \rightarrow M_I$$

We will denote by l_i the gluing map supplied by Proposition 5.5, taking $\partial_+ C(\Gamma_T^{(i)})$ to $\partial_- C(\Gamma_T^{(i+1)})$ when $1 \leq i < n$. The map l_0 takes $\partial C(\Gamma_S)$ to $\partial_- C(\Gamma_T^{(1)})$, and $l_n: \partial_+ C(\Gamma_T^{(n)}) \rightarrow \partial C(\overline{\Gamma_S}^{c^{-2n}})$.

Recall from the proof of Proposition 4.15 that for each i between 1 and n and j between 1 and 4, the annular cusp cross section $DB_j^{(i)}$ of $C(H^{(i)})$ is defined by $\phi_i \circ p_T(DB_j)$, where DB_j is as defined in Lemma 4.12 and pictured in Figure 9. Thus each of T_1 and T_2 meets each of the $C(\Gamma_T^{(i)})$ in a collection of annular cusp cross sections parallel to a subcollection of the $DB_j^{(i)}$, $j \in \{1, 2, 3, 4\}$. Similarly,

each of $T_1 \cap C(\overline{\Gamma}_S)$ and $T_2 \cap C(\overline{\Gamma}_S)$ is parallel to one of the annular cross sections \overline{A}_1 or \overline{A}_2 defined in Proposition 4.15.

It follows from the proof of Proposition 4.15 that when $1 \leq i < n$ and $a_i = 0$, $l_i = \iota_+^{(i)}(\iota_-^{(i)})^{-1}$ takes $\partial_+ DB_j^{(i)}$ to $\partial_- DB_j^{(i-1)}$ for each j between 1 and 4. Thus by the description above, if $a_i = 1$ then l_i acts on the indices j by the permutation (12)(34). The proof of Proposition 4.15 likewise implies that if $a_0 = 0$, then $l_0(\partial A_j) = \partial_- DB_j^{(1)} \sqcup \partial_- DB_{j+2}^{(1)}$ for $j = 1, 2$; hence if $a_0 = 1$, then $l_0(\partial A_j) = \partial_- DB_{3-j}^{(1)} \sqcup \partial_- DB_{5-j}^{(1)}$ for $j = 1, 2$. A similar dichotomy holds for l_n .

Remark. The definitions of the annular cusp cross-sections in Lemmas 4.10 and 4.11 depended on a particular collection of horospheres centered at the ideal vertices of \mathcal{P}_1 and \mathcal{P}_2 . These give rise to a particular collection of horospherical cross-sections of the cusps of $F^{(0)}$, which is not preserved by ϕ_{m_2} .

It would thus be more accurate to say, for example, that when $a_i = 1$ and $1 \leq i < n$, $l_i(\partial_+ DB_1^{(i)})$ is a cusp cross-section of $\partial_- C(\Gamma_T^{(i+1)})$ parallel to $\partial_- DB_2^{(i+1)}$. Since we are interested in computing moduli, and these are unaffected by similarities, we have ignored this distinction above and will continue to do so below.

Claim. For each i between 1 and n ,

$$T_1 \cap M_T^{(i)} = \begin{cases} DB_1^{(i)} \cup DB_3^{(i)} & \text{if } c_{i-1} = 0 \\ DB_2^{(i)} \cup DB_4^{(i)} & \text{if } c_{i-1} = 1. \end{cases}$$

Furthermore, $T_1 \cap \overline{M}_S = \overline{A}_1$ if $c_n = 0$ and \overline{A}_2 if $c_n = 1$.

Proof of claim. This is proved by induction on i . In the base case $i = 1$, since $c_0 = a_0$ and $T_1 \cap C(\Gamma_S) = A_1$, the conclusion in this case follows directly from the dichotomy in the behavior of l_0 recorded above the claim.

Suppose now that the claim holds for some $i < n$, and note that therefore $T_1 \cap M_T^{(i)}$ has components $DB_j^{(i)}$ and $DB_{j'}^{(i)}$, where $j, j' \in \{1, 2, 3, 4\}$ have the same parity, which is opposite that of c_{i-1} . By definition, c_i has the opposite parity from c_{i-1} if and only if $a_i = 1$. By the above, l_i acts on the indices of the $DB_j^{(i-1)}$ changing parity if and only if $a_i = 1$ as well. Hence the claim holds for $i + 1$.

Induction thus implies that the claim holds for each $i < n$. The case $i = n$ follows similarly. \square

The moduli of A_1 , A_2 , \overline{A}_1 , and \overline{A}_2 are described in Lemma 4.10, and those of the $DB_j^{(i)}$ are described in Lemma 4.12. Using these descriptions and Lemma 4.9, the claim above shows that the imaginary part of $m(T_1)$ is as described in the statement of the proposition. The description of the imaginary part of $m(T_2)$ follows similarly.

Now recall the definitions of the arcs a_1 and $db_j^{(i)}$ for $1 \leq i \leq n$ and $j = 1, 3$, and the collections of arcs \mathcal{A}_2 and $DB_j^{(i)}$ for $1 \leq i \leq n$ and $j = 2, 4$, from the proof of Proposition 4.15. For our purposes here, we additionally define \mathcal{A}_1 to be a collection of five arcs evenly spaced around A_1 , each perpendicular to ∂A_1 at each of its endpoints, such that $a_1 \in \mathcal{A}_1$. We analogously define $DB_j^{(i)}$ as a collection of evenly spaced arcs in $DB_j^{(i)}$ containing $db_j^{(i)}$ for $1 \leq i \leq n$ and $j = 1, 3$.

Claim. If $a_0 = 0$ then $l_0(\partial\mathcal{A}_j) = \partial_-DB_j^{(1)} \cup \partial_-DB_{j+2}^{(1)}$ for $j = 1, 2$, and if $a_0 = 1$ then $l_0(\partial\mathcal{A}_j) = \partial_-DB_{3-j}^{(1)} \cup \partial_-DB_{5-j}^{(1)}$. Similarly, for $1 \leq i \leq n-1$,

$$l_i(\partial_+DB_j^{(i)}) = \partial_-DB_j^{(i+1)} \text{ for } j = 1, 2, 3, 4, \text{ if } a_i = 0,$$

$$l_i(\partial_+DB_j^{(i)}) = \begin{cases} \partial_-DB_{3-j}^{(i+1)} & \text{for } j = 1, 2, \\ \partial_-DB_{7-j}^{(i+1)} & \text{for } j = 3, 4, \end{cases} \text{ if } a_i = 1.$$

Also, if $a_n = 0$ then $l_n^{-1}(\partial\bar{\mathcal{A}}_j) = \partial_+DB_j^{(n)} \cup \partial_+DB_{j+2}^{(n)}$ for $j = 1, 2$, and if $a_n = 1$ then $l_n^{-1}(\partial\bar{\mathcal{A}}_j) = \partial_+DB_{3-j}^{(n)} \cup \partial_-DB_{5-j}^{(n)}$.

In the discussion above the first claim, we recorded the analogous dichotomy to that of the claim above for the action of the gluing maps l_i on boundaries of annular cusp cross-sections. The substance of this claim is thus that the gluing maps preserve arc endpoints.

Proof of claim. Suppose first that $a_i = 0$, so by its definition $l_i = \iota_+^{(i)}(\iota_-^{(i)})^{-1}$. The proof of Proposition 4.15 directly addresses the cases of \mathcal{A}_2 , $\bar{\mathcal{A}}_2$, and $DB_j^{(i)}$, where $j = 2$ or 4 . In the remaining case of \mathcal{A}_1 , the definition implies that $\partial\mathcal{A}_1$ consists of ten points, five evenly spaced around each component of $\partial\mathcal{A}_1$, with each such collection containing a point of ∂a_1 . Also by definition, $\partial_-DB_j^{(1)}$ is a collection of five points spaced evenly around $\partial_-DB_j^{(1)}$, one of which is $\partial_-db_j^{(1)}$ for $j = 1, 3$. By the proof of Proposition 4.15, $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ takes ∂a_1 to $\partial_-db_1^{(1)} \cup \partial_-db_3^{(1)}$; hence the entire collection $\partial\mathcal{A}_1$ is taken to $\partial_-DB_1^{(1)} \cup \partial_-DB_3^{(1)}$ since $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ is an isometry. The remaining cases when $a_i = 0$, $i \geq 1$, follow similarly.

To illustrate the case $a_i = 1$ we focus on the subcase $1 \leq i < n$. When $a_0 = 1$, l_i takes $\partial_+DB_1^{(i)}$ to $\partial_-DB_2^{(i+1)}$, for example. The crucial observation here is that $l_0(\partial_+db_1^{(i)})$ is in $\partial_-DB_2^{(i+1)}$. This holds because by definition, $\partial_+db_1^{(i)}$ is a point in the edge of the triangulation Δ_T which exits the ideal vertex 1. (This is the top edge in Figure 11.) Although ϕ_{m_2} does not preserve Δ_T , it preserves this edge, exchanging its endpoints at 1 and 2. Since $\partial_-DB_2^{(i+1)}$ has a point in each edge which exits 2, it contains $\phi_{m_2}(\partial_+db_1^{(i)})$. Since the points of $\partial_+DB_1^{(i)}$ are evenly spaced around $\partial_+DB_1^{(i)}$ and the same is true for $\partial_-DB_2^{(i+1)}$, it follows that $l_0(\partial_+DB_1^{(i)}) = \partial_-DB_2^{(i+1)}$.

Since ϕ_{m_2} takes the edge of Δ_T to itself and exchanges its endpoints, $l_0(\partial_+db_3^{(i)}) \in \partial_-DB_4^{(i+1)}$ in this case. Then it follows from ‘‘even-spacedness’’ that $l_0(\partial_+DB_3^{(i)}) = \partial_-DB_4^{(i+1)}$. The same argument implies that $\partial_-db_1^{(i+1)} \in l_0(\partial_+DB_2^{(i+1)})$ and therefore that $l_0(\partial_+DB_2^{(i)}) = \partial_-DB_1^{(i+1)}$, and similarly that $l_0(\partial_+DB_4^{(i)}) = \partial_-DB_3^{(i+1)}$. The same sequence of observations, applied to $\partial\mathcal{A}_j$ and $\partial\bar{\mathcal{A}}_j$, $j = 1, 2$, completes the claim. \square

The second claim implies that the set

$$\mathcal{A}_1 \cup \mathcal{A}_2 \cup \bigcup_{i,j} DB_j^{(i)} \cup \bar{\mathcal{A}}_1 \cup \bar{\mathcal{A}}_2$$

consists of a disjoint union of closed geodesics, some in T_1 and some in T_2 , each meeting any of the geodesics $F^{(i)} \cap T_1$ or $F^{(i)} \cap T_2$ perpendicularly in up to five

points. That $m(T_1)$ and $m(T_2)$ have imaginary part equal to 0 now follows as in the proof of Proposition 4.15. \square

Proposition 6.5 allows us to describe arbitrarily large subfamilies of the M_I which have different cusp parameters.

Corollary 6.6. *For $0 \leq k \leq n$, let $I_k = (a_0, a_1, \dots, a_n)$ be defined by $a_i = 0, i \neq k$, and $a_k = 1$. The cusp parameters of M_{I_k} are not $\mathrm{PGL}_2(\mathbb{Q})$ -equivalent to those of $M_{I_{k'}}$, for $k \neq k'$, when both are less than $(n+1)/2$.*

Proof. By Proposition 6.5, the cusps of M_{I_k} have moduli described below.

$$m(T_1) = i \left[\frac{6}{5} + \frac{4}{5} (n+4k) \sqrt{2} \right] \quad m(T_2) = i \left[\frac{6}{5} + \frac{4}{5} (5n-4k) \sqrt{2} \right]$$

Since $m(T_1)$ and $m(T_2)$ are both of the form described in Lemma 4.16 for any k , if the cusp parameters of M_{I_k} are equivalent to those of $M_{I_{k'}}$, then one of the two cases below holds.

$$\begin{array}{ll} n+4k = n+4k' & \text{and} \quad 5n-4k = 5n-4k' \\ n+4k = 5n-4k' & \text{and} \quad 5n-4k = n+4k' \end{array}$$

In the first case, $k = k'$, and in the second, $k' = n - k$. Thus as long as k and $k' < (n+1)/2$ are unequal, their cusp parameters are as well. \square

There are also arbitrarily large subfamilies which share cusp parameters.

Corollary 6.7. *For $0 \leq k < n$, let $I_k = (a_0, \dots, a_n)$ be defined by $a_i = 0, i \neq k, k+1$, and $a_k = a_{k+1} = 1$. For each k , the cusp parameters of M_{I_k} are*

$$m(T_1) = i \left[2 + 4 \left(n - \frac{4}{5} \right) \sqrt{2} \right] \quad m(T_2) = i \left[\frac{2}{5} + \frac{4}{5} (n+4) \sqrt{2} \right],$$

up to the action of $\mathrm{PGL}_2(\mathbb{Q})$.

APPENDIX A. PROOF OF LEMMA 2.7

Following Morgan [20], we define a *pared manifold* to be a pair (M, P) , where M is a compact, orientable, irreducible 3-manifold with nonempty boundary which is not a 3-ball, and $P \subseteq \partial M$ is the union of a collection of disjoint incompressible annuli and tori satisfying the following properties:

- Every noncyclic abelian subgroup of $\pi_1 M$ is conjugate into the fundamental group of a component of P .
- Every map $\phi: (S^1 \times I, S^1 \times \partial I) \rightarrow (M, P)$ which induces an injection on fundamental groups is homotopic as a map of pairs to a map ψ such that $\psi(S^1 \times I) \subset P$.

This definition is intended to capture the topology of the compact manifold obtained by truncating the cusps of the convex core of a geometrically finite hyperbolic 3-manifold by open horoball neighborhoods. Indeed, Corollary 6.10 of [20] asserts that if (M, P) is obtained in this way, where P consists of the collection of boundaries of the truncating horoball neighborhoods, then (M, P) is a pared manifold.

Lemma 2.7 from the body of this paper asserts that if (M, P) has the pared homotopy type of a geometrically finite hyperbolic manifold \mathbb{H}^3/Γ where Γ is not Fuchsian and $\partial C(\Gamma)$ is totally geodesic, then $M - P$ is homeomorphic to $C(\Gamma)$. The

key point of the proof is that the geometric conditions on Γ ensure that (M, P) is an acylindrical pared manifold. Then Johannson's Theorem [11], that pared homotopy equivalences between acylindrical pared manifolds are homotopic to pared homeomorphisms, applies. We expand on this below.

It is worth noting that Lemma 2.7 fails in more general circumstances. Canary-McCullough give examples of this phenomenon in [7], where for instance they describe homotopy equivalent non-Fuchsian geometrically finite manifolds with incompressible convex core boundary which are not homeomorphic (Example 1.4.5). Their memoir [7] is devoted to understanding the ways in which homotopy equivalences of hyperbolic 3-manifolds can fail to be homeomorphic to homeomorphisms, and Lemma 2.7 follows quickly from results therein.

The treatment of Canary-McCullough itself uses the theory of *characteristic submanifolds* of manifolds with *boundary pattern* developed in [11]. The characteristic submanifold of a manifold with boundary pattern is a maximal collection of disjoint codimension-zero submanifolds, each of which is an interval bundle or Seifert-fibered space embedded reasonably with respect to the boundary pattern. Rather than attempting to establish all of the notation necessary to define this formally, we refer the interested reader to [11] and [7]. Here we simply transcribe the relevant theorem of [7], which places strong restrictions on the topology of the characteristic submanifold of a pared manifold whose boundary pattern is determined by the pared locus.

For the purposes of Lemma 2.7 we exclude from consideration certain pared manifolds which never arise from convex cores of geometrically finite hyperbolic 3-manifolds. We say (M, P) is *elementary* if it is homeomorphic to one of $(T^2 \times I, T^2 \times \{0\})$, $(A^2 \times I, A^2 \times \{0\})$, or $(A^2 \times I, \emptyset)$, where T^2 and A^2 denote the torus and annulus, respectively; otherwise (M, P) is *nonelementary*. Define $\partial_0 M := \overline{M - P}$. We say an annulus properly embedded in $M - P$ is *essential* in (M, P) if it is incompressible and boundary-incompressible in $M - P$. For a codimension-0 submanifold V embedded in M , we denote by $\text{Fr}(V)$ the *frontier* of V (that is, its topological boundary in M), and note that $\text{Fr}(V) = \partial V - (V \cap \partial M)$. With notation thus established, the following theorem combines the definition of the characteristic submanifold with Theorem 5.3.4 of [7].

Theorem. *Let (M, P) be a nonelementary pared manifold with $\partial_0 M$ incompressible. Select the fibering of the characteristic submanifold so that no component is an I -bundle over an annulus or Möbius band.*

- (1) *Suppose V is a component of the characteristic submanifold which is an I -bundle over a surface B . Then each component of the associated ∂I -bundle is contained in $\partial_0 M$, each component of the associated I -bundle over ∂B is either a component of P or a properly embedded essential annulus, and B has negative Euler characteristic.*
- (2) *Suppose V is a Seifert fibered component of the characteristic submanifold. Then V is homeomorphic either to $T^2 \times I$ or to a solid torus. If V is homeomorphic to $T^2 \times I$, then one of its boundary components lies in P , the other components of $V \cap \partial M$ are annuli in $\partial_0 M$, and all components of $\text{Fr}(V)$ are properly embedded essential annuli. If V is a solid torus, then $V \cap \partial M$ has at least one component, each an annulus either containing a component of P or contained in $\partial_0 M$. The components of $\text{Fr}(V)$ are properly embedded essential annuli.*

The characteristic submanifold contains regular neighborhoods of all components of P .

The key claim in the proof of Lemma 2.7 is a further restriction on the characteristic submanifold of (M, P) , in the case that M is obtained from the convex core of a non-Fuchsian geometrically finite manifold with totally geodesic convex core boundary by removing horoball neighborhoods of the cusps. P is the union of the boundaries of these neighborhoods.

Claim. (M, P) as above is nonelementary, and $\partial_0 M$ is incompressible. The characteristic submanifold of (M, P) consists only of (Seifert fibered) regular neighborhoods of the components of P , each of whose boundary has a unique component of intersection with ∂M .

We prove the claim below, but assuming it for now, the proof of Lemma 3 proceeds as follows. A representation as given in the statement of the lemma induces a pared homotopy equivalence between (M, P) and the pared manifold (N, Q) obtained by truncating $C(\Gamma)$ with open horoball neighborhoods. Since $C(\Gamma)$ has totally geodesic convex core boundary, (N, Q) is as described by the claim; hence (M, P) is as well (see Theorem 2.11.1 of [7], for example). Johansson's Classification Theorem (cf. [7], Theorem 2.9.10) implies that the original pared homotopy equivalence is homotopic to one which maps the complement of the characteristic submanifold of (M, P) homeomorphically to the complement of the characteristic submanifold of (N, Q) . It follows from the claim that these are homeomorphic to $M - P$ and $N - Q$, respectively, and the lemma follows.

Proof of claim. As was mentioned above, the elementary pared manifolds do not arise from geometrically finite hyperbolic manifolds. Since (M, P) is obtained from the convex core of a geometrically finite manifold with totally geodesic convex core boundary, the following are known not to occur:

- (1) A compressing disk for $\partial_0 M$.
(By definition $\partial_0 M$ lifts to a geodesic hyperplane in \mathbb{H}^3 , hence the induced map $\pi_1 \partial_0 M \rightarrow \pi_1 M$ is injective.)
- (2) An *accidental parabolic*; that is, an incompressible annulus properly embedded in M with one boundary component in P and one in $\partial_0 M$, which is not parallel to P .
(Every essential curve on $\partial_0 M$ that is not boundary-parallel is homotopic to a geodesic, but an element of $\pi_1(M)$ corresponding to an accidental parabolic has translation length 0.)
- (3) A *cylinder*; that is, a properly embedded essential annulus in $M - P$.
(The double DM of M across $\partial_0 M$ is a hyperbolic manifold and the double of a cylinder in M is an essential torus in DM .)

We show that if the characteristic manifold has any components other than those listed in the claim then at least one of the above facts cannot hold.

For a component V of the characteristic submanifold which is an I -bundle over a surface B , at least one component of the associated I -bundle over ∂B must be properly embedded, since otherwise we would have $M = V$ and it is well known that an I -bundle over a surface does not admit a hyperbolic structure with totally geodesic convex core boundary unless the convex core is a Fuchsian surface. But this annulus violates fact 2 or 3. Thus there are no I -bundle components of the characteristic submanifold.

If V is a Seifert fibered component of the characteristic submanifold homeomorphic to $T^2 \times I$, then one component of ∂V is a torus $P_1 \subset P$, and all other components of $\partial V \cap \partial M$ are annuli in $\partial_0 M$. If this second class is nonempty, then each component of $\text{Fr}(V)$ is an essential annulus properly embedded in $M - P$. This is not possible by fact 3, so $\partial V \cap \partial M$ consists only of P_1 and V is a regular neighborhood of P_1 .

If V is a solid torus and $V \cap \partial M$ contains a component of P , then a similar argument shows that this is the unique component of $\partial V \cap \partial M$, so in this case V is a regular neighborhood of an annular component of P . If on the other hand $V \cap \partial M$ does *not* contain any components of P , then it has at least two components, for otherwise a meridional disk of V determines a boundary compression of the annulus $\text{Fr}(V)$ in $M - P$. But then any component of $\text{Fr}(V)$ violates fact 3. \square

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