1. L'Hôpital's Rule says:
IF $f$ and $g$ are differentiable on an open interval containing $a$, and $g'(x) \neq 0$ near $a$ (except maybe at $a$) and $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, [that is, if $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$]
THEN $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. [Note: This works if $a$ is a real number or is $\pm\infty$].

   (a) Use L'Hôpital's Rule to evaluate $\lim_{x\to 0} \frac{\sqrt{9+3x}-3}{x}$ (as always, first check that the rule does apply here).

   (b) Use our earlier methods (multiplying by the conjuate) to evaluate $\lim_{x\to 0} \frac{\sqrt{9+3x}-3}{x}$ and check that your answer matches what you got by using L'Hôpital's Rule. (If the algebra is taking you a long time, check with me about how to write it efficiently. This is a problem type we've done many times before, so it shouldn't take very long, and is good review for the final.)

   (c) Show that the hypothesis is necessary in L'Hôpital's Rule by evaluating $\lim_{x\to 1} \frac{x^3+x^2-2x}{x-2}$ and evaluating $\lim_{x\to 1} \frac{3x^2+2x-2}{1}$. Explain why this shows that you need to check if the hypothesis holds before using L'Hôpital's Rule.

2. We will prove a special case of L'Hôpital's Rule: Along with the assumptions above, assume that $a$ is a real number, $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, $f'$ and $g'$ are continuous at $a$, $f(a) = g(a) = 0$, and $g'(a) \neq 0$. (That is, we are assuming that we start with the indeterminate form $0/0$, and that one application of l'Hopital's rule will give us an answer that is not indeterminate. The general proof is similar, but its a little more complicated to keep track of all the possibilities such as starting with different indeterminate forms, or needing to apply the rule more than once to get a determinate answer.)

   (a) Fill in the reasons in the following proof by referring to the assumptions in this problem and to properties of limits and derivatives.

*Proof.* $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$ because

$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$ because

$= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$ because

$$= \lim_{x\to a} \frac{f'(a)}{g'(a)} \text{ because}$$

□

(b) In which step do you use the assumption that we start with an indeterminate form? Would this step still be valid if $f(a)$ or $g(a)$ was a non-zero finite number?

3. Evaluate the following. In each case, decide whether L'Hôpital's rule will be helpful or whether you should consider other methods. If you see more than one method that you think will work, try both and see which is easiest. Notice that if you start with an indeterminate form other than $0/0$ or $\infty/\infty$ you can generally use properties of fractions or logs and exponents to rearrange it into one of those indeterminate forms in order to use l'Hopital's rule.

   (a) $\lim_{x\to 0} \frac{x}{e^{x^2}}$

   (b) $\lim_{x\to 0} \frac{\sin 4x}{\sin 3x}$

   (c) $\lim_{x\to 1} \frac{x^n - 1}{x - 1}$, $n$ is a positive integer

   (d) $\lim_{x\to 0} x^{1/x}$

   (e) $\lim_{x\to\infty}\left(x - \sqrt{x^2+1}\right)$

   (f) $\lim_{z\to 0} \frac{\tan 4z}{\tan 7z}$

   (g) $\lim_{x\to 0} \frac{x^2 - 2x}{3x - 2}$

   (h) $\lim_{x\to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \tan x$