1. L'Hôpital's Rule says:
IF f and g are differentiable on an open interval containing a, and $g'(x) \neq 0$ near a (except maybe at a) and $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, [that is, if $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$]

THEN $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. [Note: This works if a is a real number or is $\pm\infty$].

   (a) Use L'Hôpital's Rule to evaluate $\lim_{x\to 0} \frac{\sqrt{9+3x}-3}{x}$ (as always, first check that the rule does apply here).

   (b) Use our earlier methods (multiplying by the conjuate) to evaluate $\lim_{x\to 0} \frac{\sqrt{9+3x}-3}{x}$ and check that your answer matches what you got by using L'Hôpital's Rule.

   (c) Show that the hypothesis is necessary in L'Hôpital's Rule by evaluating $\lim_{x\to 1} \frac{x^3+x^2-2x}{x-2}$ and evaluating $\lim_{x\to 1} \frac{3x^2+2x-2}{1}$. Explain why this shows that you need to check if the hypothesis holds before using L'Hôpital's Rule.

2. Prove a special case of L'Hôpital's Rule: Along with the assumptions above, assume that a is a real number, $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, f' and g' are continuous at a, $f(a) = g(a) = 0$, and $g'(a) \neq 0$.

   (a) First, note $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$. Why is that true? (Because of which assumption?)

   (b) Write out the definitions of $f'(a)$ and $g'(a)$ using the limit definition of the derivative (we had two forms - which one seems like it would be more relevant here?).

   (c) Using the above, write out $\frac{f'(a)}{g'(a)}$ in terms of the limit definition of the derivative, and then combine the limits into one.

   (d) What can you cancel out?

   (e) Simplify to show that you get the right conclusion.

   (f) Summarize what you showed.

3. Evaluate the following. L'Hôpital's rule will probably be helpful.

   (a) $\lim_{z\to 0} \frac{\tan 4z}{\tan 7z}$

   (b) $\lim_{x\to 1} \frac{x^n-1}{x-1}$, n is a positive integer

   (c) $\lim_{x\to 0} x^{1/x}$

   (d) $\lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right)$

4. It is important to remember that, while L'Hôpital's rule is very useful, it is not a magic bullet to solve all problems. Evaluate the following.

   (a) $\lim_{x \to 0} \frac{\sin 4x}{\sin 3x}$

   (b) $\lim_{x \to 0} \frac{x^2 - 2x}{3x - 2}$

   (c) $\frac{d}{dx} \left(\frac{x^2 - 4}{x + 2} \right) \bigg|_{x=2}$

   (d) $\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2} \right) \tan x$