Calculus II ESP **Spring 2013**

Worksheet 7

Thursday, Feb 7th

Instructions: Today, we will work in groups as usual, but you will **turn in your solution to #3**. I will read them and hand them back with comments next week. Writing up clear explanations of your work is an important skill in the long run, and this kind of practice should also help when it comes time to write your work on exams. So, turning in a solution is mainly for your benefit, but will also count for your participation grade for today. You must turn in something that shows you have put some effort into solving the problem and explaining what you did.

1. Warm-up - we will discuss at least the first couple parts of this problem on the board as a class.
   (a) Write down the general rule for integration by parts. What derivative rule did this come from? How can you re-derive the integration by parts formula if you forget it?
   (b) Evaluate $\int x^2 \ln x \, dx$
   (c) Evaluate $\int \ln x \, dx$
   (d) Evaluate $\int e^{ax} \sin bx \, dx$ (a and b are constants)
2. Find the fallacy in the following argument that $0 = 1$.

   Let:

$$u = \frac{1}{x} \qquad dv = dx$$
$$du = -\frac{1}{x^2}dx \qquad v = x$$

   So integration by parts applied to $\int 1/x \, dx$ yields:

$$\begin{aligned} 0 + \int \frac{dx}{x} &= \left(\frac{1}{x}\right) x - \int \left(\frac{-1}{x^2}\right) x \, dx \\ &= 1 + \int \frac{dx}{x} \end{aligned}$$

   Thus, by canceling $\int \frac{dx}{x}$ from both sides, $0 = 1$.

3. You will **turn in** your solution to this problem at the end of class. Aim for your work to be clear enough that a classmate could read it and learn how to do the problem.

   Background: Recall that, for a natural number n, $n! = n(n-1)(n-2)(n-3) * ... * 1$ (pronounced n factorial). This is a well defined function for positive integers, but what if we wanted to extend the idea of the factorial function to a continuous function? The general process of taking a set of discrete data and extending it to get data in between those points is called **interpolation** (and does not usually yield unique answers, but you can ask that various specific conditions be satisfied, for instance getting a smooth function).

   One of the most important functions in analysis is the *gamma function*,

   $$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt = \lim_{m\to\infty} \int_0^m e^{-t} t^{x-1}\, dt, \qquad x > 0.$$

   (This definition of an integral from 0 to ∞ as a limit is an example of what we call improper integrals. They are discussed further in section 7.7 of your book.)

   (a) Use integration by parts to prove that $\Gamma(x+1) = x\Gamma(x)$. (Remember that t is the variable of integration and within the integral x is a constant!)

   (b) Show that $\Gamma(1) = 1$.

   (c) Explain why the previous two parts show that $\Gamma(n) = (n-1)!$ for all natural numbers n.

   The gamma function provides a simple example of a continuous function which **interpolates** the values of $n!$ for natural numbers n.

4. (Optional, if you have time.) The region under the curve $y = \cos x$ between $x = 0$ and $x = \pi/2$ is revolved about the y-axis. Find the volume of the resulting solid.

5. Suggested homework: look up the gamma function online and read more about it. Can you find some interesting facts about its history and applications?

Solutions To 3a

Method #1 (Apply IBP to $\Gamma(x+1)$, Choose $u = t^x$, $dv = e^{-t}dt$)

$$\Gamma(x) = \lim_{m\to\infty} \int_0^m e^{-t} t^{x-1}\,dt \quad (x>0)$$

$$\Gamma(x+1) = \lim_{m\to\infty} \underbrace{\int_0^m e^{-t} t^{x}\,dt} = \lim_{m\to\infty} \left[-t^x e^{-t}\Big|_0^m - \int_0^m -e^{-t}\, x t^{x-1}\,dx \right]$$

Note x constant wrt dt
so can move outside integral

$u = t^x \qquad dv = e^{-t}dt$

$du = xt^{x-1}dx \qquad v = -e^{-t}$

$(x = \text{const})$

$$= \lim_{m\to\infty} \left[\overbrace{-m^x e^{-m}}^{A} + \cancel{0^x e^0}^{\,0} + x \int_0^m e^{-t} t^{x-1}\,dx \right]$$

$\left(\lim_{m\to\infty} A = 0\right)$

Note $\frac{\infty}{\infty}$ indeterminate, but
$\lim_{m\to\infty} A = 0$ b/c exponential functions
grow faster than polynomials.

Challenge: Give a careful proof that
$\lim_{x\to\infty} \frac{e^{-m}}{m^x} = 0$, keeping in mind that
x might not be
a whole number.

$$= \lim_{m\to\infty} x \int_0^m e^{-t} t^{x-1}\,dx$$

$$= x\,\Gamma(x) \quad \checkmark$$

Method 2 (Apply IBP to $\Gamma(x+1)$, Choose $u = e^{-t}$, $dv = t^{x-1}dt$)

Note x constant wrt dt
so can move outside integral

$$\Gamma(x) = \lim_{m\to\infty} \int_0^m e^{-t} t^{x-1}\,dt \quad (x>0)$$

$$\Gamma(x+1) = \lim \underbrace{\int^m e^{-t} t^{x}\,dt} = \underbrace{\left[e^{-t} \cdot \frac{t^{x+1}}{x+1} \right]_0^{\infty}}_{\to\, 0} + \frac{1}{x+1} \underbrace{\int_0^{\infty} e^{-t} t^{x+1}\,dt}_{\Gamma(x+2)}$$

$u = e^{-t} \qquad dv = t^x dt$

$du = -e^{-t}dt \qquad v = \frac{t^{x+1}}{x+1}$

by the same logic
as in prev method.

So $\Gamma(x+1) = \frac{1}{x+1} \cdot \Gamma(x+2) \Rightarrow (x+1)\cdot\Gamma(x+1) = \Gamma(x+2)$

Since we showed this for all positive numers x, it is equivalent to $x\Gamma(x) = \Gamma(x+1)$
(replace x by $x-1$)

Method 3 (Apply IBP to $\Gamma(x)$)

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \qquad u = e^{-t} \qquad dv = t^{x-1}\, dt$$

$$du = -e^{-t}\, dt \qquad v = \frac{t^x}{x}$$

$$= \underbrace{\left[\frac{1}{x} t^x e^{-t}\right]_0^\infty}_{0} + \frac{1}{x} \underbrace{\int_0^\infty e^{-t} t^x\, dt}_{\Gamma(x+1)}$$

So $\Gamma(x) = \frac{1}{x}\Gamma(x+1)$

So $\Gamma(x+1) = x\Gamma(x)$

as in method #1

Note x constant wrt dt
so can move outside integral.

Note: these methods have gotten shorter because I have shown less detailed work when steps are the same as what is done above, more than due to any fundamental difference in how hard each method is.