Calculus II ESP

Improper Integrals

Worksheet # 12

Spring 2013

We call a definite integral "improper" if it is either evaluated over an infinite interval, or if the integrand is unbounded within the interval of evaluation.

1. Improper Integrals over Infinite Intervals. If one endpoint of a definite integral is $\pm\infty$, we can evaluate the integral by replacing that endpoint with a placeholder variable, and taking the limit as that variable goes to $\pm\infty$. So, we define $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$, and we have a similar definition if the upper limit is finite and the lower limit is $-\infty$. (Remember, we did an example of this on a worksheet a couple weeks ago with the Gamma function).

   (a) Evaluate $\int_1^\infty \frac{1}{x}dx$. (Remember to always show your work with limits written out clearly.)

   (b) Evaluate $\int_1^\infty \frac{1}{x^p}dx$ for $p > 1$. (If this is confusing, first take $p = 2$ and then think about what will happen in general for $p > 1$).

   (c) Evaluate $\displaystyle\int_1^\infty xe^{-x}\,dx$

2. Evaluating Improper Integrals by Comparison. Let $f(x)$ and $g(x)$ be positive functions on an interval $[a, b]$ (where a and/or b could be infinite).

   (a) If $f(x) < g(x)$ on the whole interval $[a, b]$, what can we say about the relationship between $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$?

   (b) Consider the two functions $f(x) = \dfrac{17(x+1)^3 e^{\sqrt{x}}}{x}$ and $g(x) = \dfrac{1}{x}$. Use what you know about $\int_1^\infty g(x)dx$ to evaluate $\int_1^\infty f(x)dx$.

   (c) Now consider $h(x) = \dfrac{1}{x^2\sqrt{x^2+5}}$ and $k(x) = \dfrac{1}{x^2}$. Using what you know about $\int_1^\infty k(x)dx$, what can you say about $\int_1^\infty h(x)dx$? (Was the comparison method as useful here as in the previous example?)

3. Improper Integrals with Unbounded Integrand. If $f(x)$ is continuous on a finite interval $(a, b]$, with $lim_{x\to a^+} = \pm\infty$, then we define $\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$. We have a similar definition if $f(x)$ has an asymptote at b, replacing b with a limit instead of a.

   (a) Evaluate $\displaystyle\int_0^{\pi/2} \tan\theta\, d\theta$

   (b) Evaluate $\displaystyle\int_1^{11} \frac{dx}{x^2+2x+1}$

4. What is wrong with the following argument? Find the correct value of the integral, or show that it diverges.

$$\int_{-1}^{2} \frac{dt}{t^2} = \left. \frac{-1}{t} \right|_{-1}^{2} = -\frac{1}{2} - \left(-\frac{1}{-1} \right) = -\frac{1}{2} - 1 = -\frac{3}{2}$$

Note: On Thursday, we will do an interesting example which will use the ideas from today's worksheet. So, be sure you understand them, and if you have any questions come to my office hours and we can discuss them.