Calculus II ESP | **Spring 2013**

Introduction to Sequences and Infinite Series

Worksheet # 14

1. (a) Give definitions and examples for the following terms: sequence, recurrence relation for a sequence, implicit formula for a sequence, explicit formula for a sequence, infinite series, convergence of a sequence, convergence of an infinite series.

   (b) Can you think of a physical example that shows that an infinite sequence can converge?

2. Write out the first 5 terms in each sequence. Then find the value of the series $\sum_{k=1}^{5} a_k$. If you think the sequence converges, make a conjecture about its limit.

   (a) $a_n = \frac{n}{n+1}$

   (b) $a_n = (-1)^n \cdot \frac{1}{n}$

   (c) $a_n = (\frac{2}{3})^n$

3. Consider the following recurrence relations. For each one: (i) Find the terms a_0, a_1, a_2 of the sequence. (ii) Find an explicit formula for the nth term of the sequence. (iii) Using a calculator, make a table with at least 10 terms and determine a plausible value for the limit of the sequence or state that it does not exist.

   (a) $a_{n+1} = a_n + 2;\ a_0 = 3$

   (b) $a_{n+1} = \frac{1}{2}a_n;\ a_0 = 1$

4. Consider the following infinite series. For each,

   (i) Find the first four terms of the sequence of partial sums.

   (ii) Use the results of the previous part to propose a formula for S_n, the nth partial sum.

   (iii) Propose a value of the series.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)}$

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$ (d) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{3^k}$

5. (Note: if we don't finish this problem today, it will appear on the next worksheet. So in that case, think about this problem and consider doing some of the suggested reading below before the next class.)

   The expression

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}$$

   where the process continues indefinitely, is called a *continued fraction*.

   (a) Show that this expression can be built in steps using the recurrence relation $a_0 = 1$; $a_{n+1} = 1 + 1/a_n$ for $n = 0, 1, 2, \ldots$. Explain why the value of the expression can be interpreted as $\lim_{n\to\infty} a_n$.

   (b) Evaluate the first five terms of the sequence $\{a_n\}$.

   (c) Using computation, and/or graphing, estimate the limit of the sequence.

   (d) Assuming the limit exists, determine it exactly (assume the limit exists, call it L, and then use the properties it must have from above to determine the value of L). Compare your solution to the golden mean, $(1 + \sqrt{5})/2$.

   (e) Further suggested reading:

      i. Research the Golden Ratio (also called the Golden Mean). Start with the wikipedia article about it. Where does it appear in nature? In other areas of math? This is a surprisingly deep area of math, you can get lost reading about it.

      ii. Research continued fractions. For some cool patterns, read the wikipedia article for generalized continued fraction expansions of π.