Calculus II ESP | **Spring 2013**

Sequences

Worksheet # 15

1. The expression

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}$$

where the process continues indefinitely, is called a *continued fraction*.

(a) Show that this expression can be built in steps using the recurrence relation $a_0 = 1$; $a_{n+1} = 1 + 1/a_n$ for $n = 0, 1, 2, \ldots$. Explain why the value of the expression can be interpreted as $\lim_{n\to\infty} a_n$.

(b) Evaluate the first five terms of the sequence $\{a_n\}$.

(c) Using computation, and/or graphing, estimate the limit of the sequence.

(d) Assuming the limit exists, determine it exactly (assume the limit exists, call it L, and then use the properties it must have from above to determine the value of L). Compare your solution to the golden mean, $(1+\sqrt{5})/2$.

(e) Further suggested reading:

i. Research the Golden Ratio (also called the Golden Mean). Start with the wikipedia article about it. Where does it appear in nature? In other areas of math? This is a surprisingly deep area of math, you can get lost reading about it.

ii. Research continued fractions. For some cool patterns, read the wikipedia article for generalized continued fraction expansions of π.

2. Find the limit of the following sequences, or determine that the limit does not exist.

(a) $\left\{\dfrac{n^{12}}{3n^{12}+4}\right\}$ (b) $\{n\sin(6/n)\}$

(c) $\left\{\dfrac{n^{100}}{n^n}\right\}$ (d) $\left\{\dfrac{n+\sin^3 n}{n+1}\right\}$

(e) $\{b_n\}$ if $b_n = \begin{cases} n/(n+1) & \text{if } n \le 5000 \\ ne^{-n} & \text{if } n > 5000 \end{cases}$

3. Pick two positive numbers a_0 and b_0 with $a_0 > b_0$ and write out the first few terms of the sequences $\{a_n\}$ and $\{b_n\}$:

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad \text{for } n = 0, 1, \ldots.$$

(Recall that the arithmetic mean $A = (p+q)/2$ and the geometric mean $G = \sqrt{pq}$ of two positive numbers p and q satisfy $A \geq G$. Verify this important inequality for a few numerical examples.)

(a) Show that $a_n > b_n$ for all n.

(b) Show that $\{a_n\}$ is a decreasing sequence and $\{b_n\}$ is an increasing sequence.

(c) Conclude that $\{a_n\}$ and $\{b_n\}$ converge.

(d) Show that $a_{n+1} - b_{n+1} < (a_n - b_n)/2$ and conclude that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. The common value of these limits is called the arithmetic-geometric mean of a_0 and b_0, denoted $\mathrm{AGM}(a_0, b_0)$. (Hint: the conclusion that the limits are equal requires the fact that if $a_n > b_n$ for all n, and $\{a_n\}$ converges to A and $\{b_n\}$ converges to B, then $A \geq B$. You do not have to prove this, but try to convince yourself that it is true.)

(e) Estimate $\mathrm{AGM}(12, 20)$. Estimate Gauss's constant $1/\mathrm{AGM}(1, \sqrt{2})$.