Calculus II ESP | **Spring 2013**

More Sequences and Series

Worksheet # 17

1. Suppose you can make an investment that will pay you a certain amount annually to you and your heirs. Suppose also that a payment of x dollars next year has a current value of βx, where $\beta \in (0, 1)$. (β is called a "discount factor" - it means that money in the future is less valuable than money today.)

   (a) Write a recurrence relation for the sequence of current values of an infinite stream of payments of x dollars each year.

   (b) Write an explicit formula for the nth term of the sequence.

   (c) Consider the infinite sum given by summing the terms of your sequence. What does it represent?

   (d) Take $x = 100$ and $\beta = .9$. What does the infinite sum approach?

2. Think of this question as a fun puzzle! There is an easy way and a hard way to solve it (try to come up with both), and it comes with a story/joke about a famous mathematician that I will tell you afterwards. Then you can try this on your friends from your other Calc 2 discussions and see which solution they come up with:

   Two trains start 100 miles apart, and head towards each other, each one going 10 mph. At the same instant, a bee leaves the first train and flies at 20 mph to the second. When it gets there, it immediately turns around and heads back to the first. Then it repeats, going back and forth between the two trains (assume the bee is always traveling at a constant speed of 20 mph, so any time it takes to turn around is negligible). By the time the trains reach each other, how much distance will the bee have covered?

3. The precise definition of a limit of a sequence: $lim_{n\to\infty} a_n = L$ if for every small number $\epsilon > 0$, there is a number N (depending on epsilon, so we could write $N = N(\epsilon)$) such that whenever $n > N$, $|a_n - L| < \epsilon$. (Recall that we did this in Calc 1 for functions.)

   (a) Use the precise definition of the limit to prove that the limit of the sequence $\{1/n\}$ is 0.

   (b) Use the precise definition of the limit to prove that the limit of the sequence $\{\frac{2n+1}{n}\}$ is 2.

4. Pick two positive numbers a_0 and b_0 with $a_0 > b_0$ and write out the first few terms of the sequences $\{a_n\}$ and $\{b_n\}$:

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad \text{for } n = 0, 1, \ldots.$$

(Recall that the arithmetic mean $A = (p+q)/2$ and the geometric mean $G = \sqrt{pq}$ of two positive numbers p and q satisfy $A \geq G$. Verify this important inequality for a few numerical examples.)

(a) Show that $a_n > b_n$ for all n.

(b) Show that $\{a_n\}$ is a decreasing sequence and $\{b_n\}$ is an increasing sequence.

(c) Conclude that $\{a_n\}$ and $\{b_n\}$ converge.

(d) Show that $a_{n+1} - b_{n+1} < (a_n - b_n)/2$ and conclude that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. The common value of these limits is called the arithmetic-geometric mean of a_0 and b_0, denoted $\mathrm{AGM}(a_0, b_0)$. (Hint: the conclusion that the limits are equal requires the fact that if $a_n > b_n$ for all n, and $\{a_n\}$ converges to A and $\{b_n\}$ converges to B, then $A \geq B$. You do not have to prove this, but try to convince yourself that it is true.)

(e) Estimate $\mathrm{AGM}(12, 20)$. Estimate Gauss's constant $1/\mathrm{AGM}(1, \sqrt{2})$.