Calculus II ESP

More Series and Alternating Series

Worksheet # 20

Spring 2013

1. Prove that if $\sum a_k$ is a convergent series of positive terms, then $\sum a_k^2$ also converges.

2. Give an example of a series that converges conditionally but not absolutely. Also give an example of an absolutely convergent series.

3. Find the number of terms of $\sum_{k=1}^{\infty} \frac{(-1)^n}{k^p}$ that must be taken to ensure that the remainder is less that 10^{-4}, for $p = 2$ and $p = 3$.

4. Decide whether the series converge absolutely, conditionally or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\pi n/2)}{n^2}$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$

5. Conditionally convergent series can behave strangely. Unlike finite sums or absolutely convergent series, rearranging the terms can change the value of the sum. We will explore this bizarre property.

   (a) Show that the series

   $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

   is convergent but not absolutely convergent (that is, it is conditionally convergent). It can be shown (we will see how in Chapter 9) that

   $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2 \qquad (*)$$

   (b) Multiply equation (*) on both sides by $\frac{1}{2}$. Now add these two series together, matching terms to show that

   $$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + + - \cdots = \frac{3}{2} \ln 2 \qquad (**)$$

(c) What do you notice about the series in (*) and (**)?

Note that we have thus rearranged the terms of a conditionally convergent series so that it converges to a different value than the original series. This is a strange property of conditionally convergent series, and why in general you must be careful about rearranging terms. Is there the same problem with an absolutely convergent series?

In fact, there is a theorem called the Riemann series theorem, which says that if a series is conditionally convergent, then its terms can be rearranged so that it converges to any given value, or even diverges.