Calculus II ESP | **Spring 2013**

Taylor Series
Worksheet # 24

1. For each of the following functions
   (i) Find the first four nonzero terms of the function's Maclaurin series.
   (ii) Write the power series using summation notation.
   (iii) Determine the interval of convergence.

   (a) $f(x) = (1 + 2x)^{-1}$ (b) $g(x) = \tan^{-1} x$

2. Find the Taylor series centered at zero for the following functions by using known Taylor series (see Table 9.5, p. 620).

   (a) $f(x) = b^x$ (b) $f(x) = \sqrt{1 - x^2}$ (c) $f(x) = \cos(2x) + 2 \sin x$

3. It is possible to have a function f which has a Taylor series that does not converge to f on its interval of convergence. Here is an example of this.

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Use the definition of the derivative to show that $f'(0) = 0$.
   (b) Assume the fact that $f^{(k)}(0) = 0$ for $k = 1, 2, 3, \ldots$. (You can prove this formally using the definition of the derivative. Feel free to try, but finish the next part first.)
   (c) Write the Taylor series for f centered at zero. What is its radius of convergence? Explain why the Taylor series for f does not converge to f for $x \neq 0$.
   (d) Show that this function fails the necessary and sufficient condition in Theorem 9.7 (p. 618) for a convergence of a Taylor series to its function.

4. An essential function in statistics and the study of the normal distribution (bell curve) is the error function:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$

   (Recall that e^{-t^2} was our classic example of a function that cannot be anti-differentiated using elementary functions.)

   (a) Compute the derivative of erf(x)
   (b) Expand e^{-t^2} in a Maclaurin series, then integrate to find the first four nonzero terms of the Maclaurin series for erf.
   (c) Use the polynomial in part (b) to approximate erf(0.15) and erf(-0.09).