Calculus II ESP

Final Exam Review

Worksheet # 28

Spring 2013

1. Find the volume generated by revolving about the y axis the region bounded by $y = \sqrt{x}$, the x axis, $x = 0$ and $x = 4$.

2. Find the limit as $n \to \infty$. Justify your answer.

   (a) $a_n = \dfrac{\ln n}{\ln(n+1)}$

   (b) $b_n = \dfrac{\cos n + n^2}{3n^2 + 4}$

3. Evaluate the following integrals or state that they diverge.

   (a) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

   (b) $\displaystyle\int_2^\infty \frac{dx}{x\ln x}$

4. Evaluate $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx$

5. Determine whether each series converges. Justify your answer.

   (a) $\displaystyle\sum \frac{k!}{e^k}$

   (b) $\displaystyle\sum \frac{(-1)^k k!}{e^k}$

6. Find a power series centered at zero for each of the following. Give the interval of convergence for your series.

   (a) $g(x) = \cos x - \sin x$

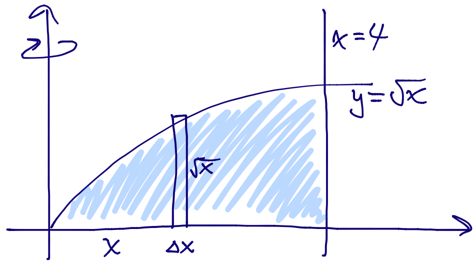
   (b) $f(x) = \ln(1 + x^2)$

7. (a) Sketch the graph of $r = sin(3\theta)$

   (b) Find the area of the region enclosed by $r = sin(3\theta)$

Solutions

(#1) $y=\sqrt{x}$, x-axis, $x=0$, $x=4$, rotate about y-axis

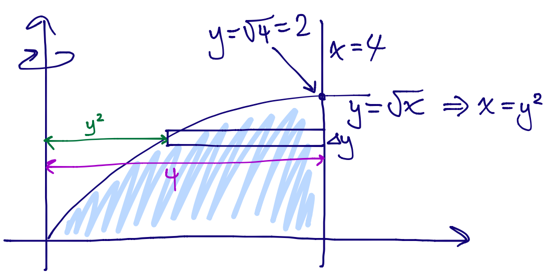


Shells: $V = 2\pi \int_0^4 x\sqrt{x}\, dx$

$= 2\pi \int_0^4 x^{3/2}\, dx$

$= 2\pi \left[\frac{2}{5} x^{5/2}\right]_0^4 = \frac{4\pi}{5}\left[4^{5/2} - 0^{5/2}\right] = \boxed{\frac{4\pi}{5}\cdot 2^5}$

OR, Washers:



$V = \pi \int_0^2 4^2 - (y^2)^2\, dy$

$= \pi \int_0^2 16 - y^4\, dy$

$= \pi \left[16y - \frac{y^5}{5}\right]_0^2 = \pi\left[2^5 - \frac{2^5}{5}\right] - 0 = \boxed{\pi\left(\frac{4}{5}\cdot 2^5\right)}$

(#2) (a) $\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} \overset{l'h}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n}$

$= 1 + 0 = \boxed{1}$

(#2b) $\lim_{n\to\infty} \frac{\cos(n)+n^2}{3n^2+4} = \underbrace{\lim_{n\to\infty} \frac{\cos(n)}{3n^2+4}}_{\downarrow\, 0} + \underbrace{\lim_{n\to\infty} \frac{n^2}{3n^2+4}}_{\downarrow\, \frac{1}{3}}$

$\frac{1}{3}$ (ratio of lead coeff) (or l'hopital twice)

By squeeze Thm, or by 2 applications of l'Hopital's rule

Squeeze Thm:

$-1 \le \cos(n) \le 1 \Rightarrow \frac{-1}{3n^2+4} \le \frac{\cos(n)}{3n^2+4} \le \frac{1}{3n^2+4}$ $(n>0)$

$\frac{-1}{3n^2+4} \xrightarrow{n\to\infty} 0$, $\frac{1}{3n^2+4} \xrightarrow{n\to\infty} 0$

So $\lim_{n\to\infty} \frac{\cos n}{3n^4+4} = 0$

So $\lim_{n\to\infty} b_n = 0 + \frac{1}{3} = \boxed{\frac{1}{3}}$

(#3) (a) $\int x e^{-x^2}\,dx = \frac{-1}{2}\int e^u\,du = \frac{-1}{2}e^u = \frac{-1}{2}e^{-x^2} + C$

let $u = -x^2$

$du = -2x\,dx$

So $\int_0^\infty x e^{-x^2}\,dx = \lim_{t\to\infty} \int_0^t x e^{-x^2}\,dx$

$= \lim_{t\to\infty} \left[\frac{-1}{2}e^{-x^2}\right]_0^t = \lim_{t\to\infty} \left[\frac{-1}{2}\underbrace{e^{-t^2}}_{\downarrow\, 0} + \frac{1}{2}e^0\right] = 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$

(#3b) $\int \frac{dx}{x\ln(x)} = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln(x)| + C$

let $u = \ln(x)$
$du = \frac{1}{x}dx$

So $\int_2^\infty \frac{dx}{x\ln(x)} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\ln(x)} = \lim_{t\to\infty} \left[\ln(\ln(x))\right]_2^t$

$= \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln(2))$ $\boxed{\text{Diverges to } \infty}$

(#4) $\int \frac{1+e^x}{1-e^x}\,dx = \int \frac{(1+e^x)}{(1-e^x)\,e^x} \underbrace{e^x\,dx}_{=du}$ $\quad u = e^x$, $du = e^x dx$

I $= \int \frac{(1+u)}{(1-u)u}\,du$

Partial Fractions:

$$\frac{1+u}{(1-u)(u)} = \frac{A}{1-u} + \frac{B}{u} \Rightarrow 1+u = Au + B(1-u)$$

$u = 0 \Rightarrow 1 = B$
$u = 1 \Rightarrow 2 = A$

So $I = \int \frac{2}{1-u}\,du + \int \frac{1}{u}\,du = -2\ln|1-u| + \ln|u| + C$
$= \boxed{-2\ln|1-e^x| + \ln|e^x| + C}$

(#5a) $\sum \frac{k!}{e^k}$ Ratio Test:

$$\lim_{k\to\infty}\left|\frac{(k+1)!}{e^{k+1}}\cdot\frac{e^k}{k!}\right| = \lim_{k\to\infty}\left|\frac{(k+1)\cdot \cancel{e^k}}{e\cdot \cancel{e^k}}\right| = \lim_{k\to\infty}\left|\frac{k+1}{e}\right| = \infty$$

So $\sum \frac{k!}{e^k}$ $\boxed{\text{diverges}}$

(#5b) $\sum \frac{(-1)^k \cdot k!}{e^k}$ Alternating Series Test:

$\lim\limits_{k\to\infty} \frac{k!}{e^k}$ Factorials grow more quickly than exponentials, so $\lim\limits_{k\to\infty}\frac{k!}{e^k} \neq 0$, so by AST the series diverges.

(So we could have also used the divergence test in part (a))

(#6a) $g(x) = \cos x - \sin x$

We know $\cos x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, $\sin(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ (for all x)

So $g(x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$

$$= \boxed{1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \dots \quad \text{(conv for all } x\text{)}}$$

(#6b) $f(x) = \ln(1+x^2)$

$$\frac{1}{1-x} = \sum x^n \quad |x|<1 \quad \text{geo series}$$

$$\frac{1}{1+x} = \sum (-1)^n x^n \quad |x|<1$$

$$\frac{1}{1+x^2} = \sum (-1)^n x^{2n} \quad |x|<1$$

$$f(x) = \ln(1+x^2)$$

$$f'(x) = 2x \cdot \frac{1}{1+x^2} = 2x \cdot \sum (-1)^n x^{2n} = 2\sum (-1)^n x^{2n+1}$$

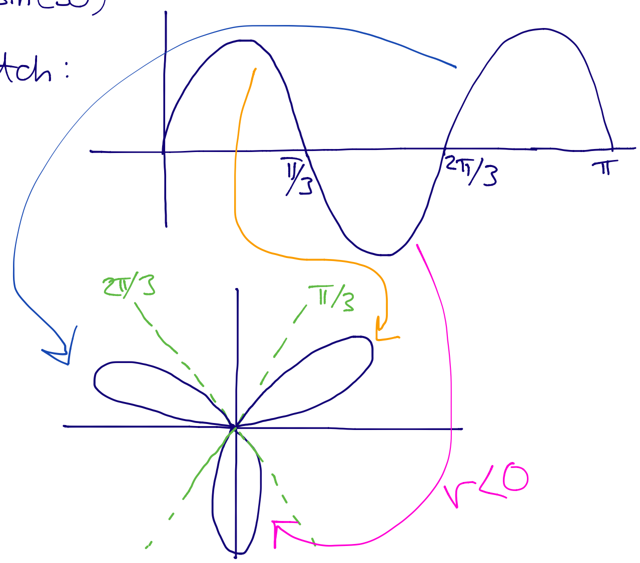
So $f(x) = \int 2\sum (-1)^n x^{2n+1} = \sum 2(-1)^n \cdot \int x^{2n+1}\,dx = \sum 2(-1)^n \cdot \frac{x^{2n+2}}{2n+2}$

$$= \boxed{\sum \frac{(-1)^n x^{2n+2}}{n+1} \quad \text{for } |x|<1}$$

(#7a) $r = \sin(3\theta)$

Cartesian Sketch:

Polar Sketch:



(#7b) Area of one petal: $\frac{1}{2}\int_0^{\pi/3} (\sin(3\theta))^2 \, d\theta$.

$$= \frac{1}{4}\int_0^{\pi/3} (1-\cos 6\theta)\, d\theta = \frac{1}{4}\theta - \frac{1}{24}\sin(6\theta)\Big]_0^{\pi/3} =$$

$$= \frac{\pi}{12} - 0 - 0 + 0 = \frac{\pi}{12}$$

So total area $= \frac{3 \cdot \pi}{12} = \boxed{\frac{\pi}{4}}$