Calculus II ESP | **Spring 2013**

Exam 2 Review

Worksheet # 21

1. Determine whether the following integrals converge, and if so, evaluate.

   (a) $\displaystyle\int_1^\infty x^2 e^{-x}\, dx$

   (b) $\displaystyle\int_0^1 \frac{x}{x-1}\, dx$

(a) $\displaystyle\int x^2 e^{-x} dx = uv - \int -2xe^{-x} = uv + 2\underbrace{\int xe^{-x} dx}_{(\star)}$

IBP: $u = x^2$ $\quad dv = e^{-x}$

$du = 2x\,dx$ $\quad v = -e^{-x}$

IBP: $z = x$ $\quad dw = e^{-x}$

$dz = dx$ $\quad w = -e^{-x}$

$(\star) = zw - \int -e^{-x} dx = zw - e^{-x}$

So $\displaystyle\int x^2 e^{-x} dx = -x^2 e^{-x} + 2\left(-xe^{-x} - e^{-x}\right) + C$

$\displaystyle = \frac{-x^2}{e^x} - \frac{2x}{e^x} - \frac{2}{e^x} + C$

So $\displaystyle\int_1^\infty x^2 e^{-x} dx = \lim_{t\to\infty} \int_1^t x^2 e^{-x} dx$

$\displaystyle = \lim_{t\to\infty} \left[\underbrace{\frac{-t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t}}_{\to 0} + \frac{1}{e} + \frac{2}{e} + \frac{2}{e} \right]$

(Since exponentials grow faster than polynomials, or by l'hôpital's rule.)

$= \boxed{\dfrac{5}{e}}$

(This limit is the kind of situation where you can come back and show more work if you have time at the end of the test.)

(b)

$$\begin{array}{r|l} & 1 + \frac{1}{x-1} \\ \hline x-1 & x \\ & x-1 \\ \hline & 0 \quad 1 \end{array}$$

$$\frac{x}{x-1} = 1 + \frac{1}{x-1}$$

So $\int \frac{x}{x-1} = \int 1 + \frac{1}{x-1}\,dx$

$= x + \ln|x-1| + C$

$\frac{x}{x-1}$ has asymptote at $x=1$

$$\int_0^1 \frac{x}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{x}{x-1}\,dx = \lim_{t\to 1^-}\left[\underset{\downarrow\,1}{t} + \underset{\downarrow\,-\infty}{\ln|t-1|} - 0 - \overset{0}{\cancel{\ln|-1|}}\right]$$

Diverges to $-\infty$

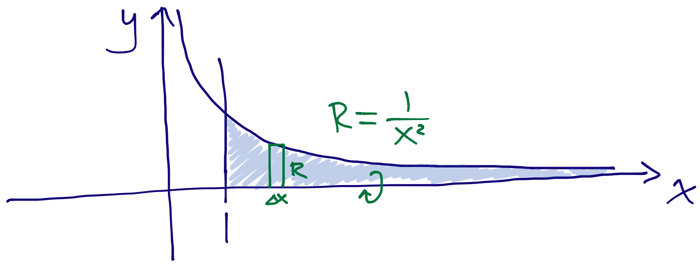
$$\boxed{\text{So } \int_0^1 \frac{x}{x-1}\,dx \text{ is divergent.}}$$

(Another Method for integrating $\int \frac{x}{x-1}\,dx$ Let $u = x-1 \Rightarrow x = u+1$
$du = dx$

$$\int \frac{x}{x-1}\,dx = \int \frac{u+1}{u}\,du = \int 1 + \frac{1}{u}\,du = u + \ln|u| + C$$
$$= \underbrace{x-1}_{} + \ln|x-1| + C$$

(-1 can be subsumed in the $+C$ above, so these answers are equivalent.))

2. Compute the volume of the solid generated by revolving the region under the graph of $y = 1/x^2$, $1 \le x \le \infty$ about the x axis.



$$V = \pi \int_1^\infty \left(\frac{1}{x^2}\right)^2 dx = \pi \lim_{t\to\infty} \int_1^t x^{-4}\,dx = \pi \lim_{t\to\infty} \left[\frac{x^{-3}}{-3} \Big|_1^t \right]$$

$$= \pi \lim_{t\to\infty} \left[\frac{-1}{3} \cdot \underbrace{\frac{1}{t^3}}_{\downarrow\, 0} + \frac{1}{3} \right] = \boxed{\frac{\pi}{3}}$$

3. Find the values of the series.

(a) $\sum_{k=0}^{\infty} (\frac{2}{3})^k$

(b) $\sum_{k=0}^{\infty} \frac{4^{k-2}+1}{5^k}$

(c) $\sum_{k=1}^{\infty} \frac{-1}{k^2+k}$

Geometric Series:

$$\sum_{k=0}^{\infty} r^k = \begin{cases} \frac{1}{1-r} & |r|<1 \\ \text{divergent} & |r|\geq 1 \end{cases}$$

(a) $$\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{3}{3}-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = \boxed{3}$$

$\left(\left|\frac{2}{3}\right|<1\right)$

(b) $$\sum_{k=0}^{\infty} \frac{4^{k-2}+1}{5^k} = \sum_{k=0}^{\infty} \left(\frac{4^{k-2}}{5^k} + \frac{1}{5^k}\right) = \sum_{k=0}^{\infty} \left(\frac{4^k}{4^2 \cdot 5^k} + \frac{1}{5^k}\right)$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{16} \cdot \left(\frac{4}{5}\right)^k + \left(\frac{1}{5}\right)^k\right) = \frac{1}{16} \cdot \frac{1}{1-\frac{4}{5}} + \frac{1}{1-\frac{1}{5}}$$

$$= \frac{1}{16} \cdot 5 + \frac{5}{4} = \frac{5+20}{16} = \boxed{\frac{25}{16}}$$

(c) $\frac{1}{k^2+k} = \frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$

$\Rightarrow 1 = A(k+1) + B(k)$

$k=0 \Rightarrow A=1$

$k=-1 \Rightarrow B=-1$

So $\frac{1}{k^2+k} = \frac{1}{k} - \frac{1}{k+1}$

So $\sum_{k=1}^{\infty} \frac{-1}{k^2+k} = -\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = -\lim_{n\to\infty}\left[\overbrace{\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)}^{(n^{\text{th}}\text{ partial Sum})}\right]$

$= -\lim_{n\to\infty}\left[1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{-1}{n}} + \cancel{\frac{1}{n}} - \frac{1}{n+1}\right]$

$= -\lim_{n\to\infty}\left[1 - \underset{\substack{\downarrow \\ 0}}{\frac{1}{n+1}}\right] = \boxed{-1}$ (Telescoping Series with tail approaching 0.)

4. Show the convergence or divergence of each series.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}+1}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

(a) $\underset{(k\geq 1)}{k^{3/2}+1 \geq k^{3/2}} > 0 \Rightarrow \frac{1}{k^{3/2}+1} \leq \frac{1}{k^{3/2}}$

So $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}+1} \leq \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ p-series $p = 3/2 > 1$ So converges

So $\boxed{\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}+1} \text{ converges}}$ by the comparison test.

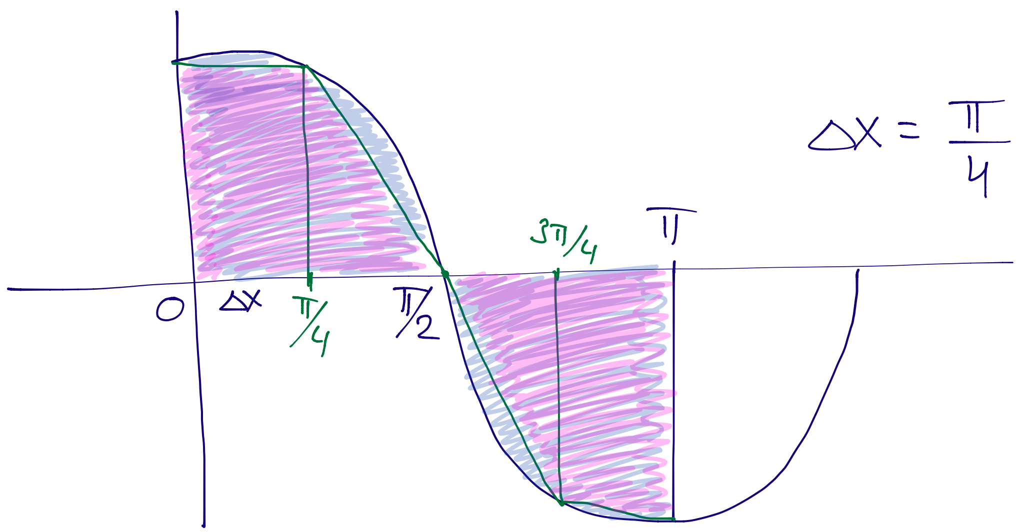
(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$ Ratio Test

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^2}{2^{k+1}} \cdot \frac{2^k}{k^2}\right| = \left|\frac{1}{2} \cdot \frac{(k^2+2k+1)}{k^2}\right| \xrightarrow{k\to\infty} \frac{1}{2} < 1$$

(Polynomials of same degree, so look at ratio of lead coefficients. Or, apply l'hopital's rule twice.)

So by the ratio test, $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$ converges.

5. Use the trapezoid rule with four subintervals to approximate $\int_0^{\pi} \cos\theta \, d\theta$.



$\Delta x = \frac{\pi}{4}$

$$T_4 = \frac{\pi}{4}\left(\frac{\cos(0) + \cos(\pi/4)}{2}\right) + \frac{\pi}{4}\left(\frac{\cos(\pi/4) + \cos(\pi/2)}{2}\right)$$

$$+ \frac{\pi}{4}\left(\frac{\cos(\pi/2) + \cos(3\pi/2)}{2}\right) + \frac{\pi}{4}\left(\frac{\cos(3\pi/4) + \cos(\pi)}{2}\right)$$

$$= \frac{\pi}{4}\left[\cancel{\frac{1 + \frac{\sqrt{2}}{2}}{2}} + \cancel{\frac{\frac{\sqrt{2}}{2} + 0}{2}} + \cancel{\frac{0 + -\frac{\sqrt{2}}{2}}{2}} + \cancel{\frac{-\frac{\sqrt{2}}{2} + -1}{2}}\right] = \boxed{0}$$

(You could find this answer faster by thinking about symmetry, but the question is not just asking for the answer - it is asking you to show you understand the trapezoid rule as well. So, your solution should include at least some work showing that you know how to use the trapezoid rule.)

6. Suppose that for an infinite series $\sum_{k=1}^{\infty} a_k$, the n^{th} partial sum is given by $S_n = 1 - \frac{1}{n}$.

(a) What is $\lim_{n\to\infty} S_n$?

(b) Does $\sum_{k=1}^{\infty} a_k$ converge or diverge? Explain your answer.

(a) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n} = 1 - 0 = \boxed{1}$

(b) $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = 1$ (by definition of a convergent series)

So $\boxed{\sum_{k=1}^{\infty} a_k \text{ converges (to 1)}}$

7. Use the precise definition of the limit to prove that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Let $\varepsilon > 0$. Want N such that $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$ when $n > N$.

$$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} < \varepsilon \iff \frac{1}{\varepsilon} < \sqrt{n} \iff \frac{1}{\varepsilon^2} < n$$

So take $\boxed{N = \frac{1}{\varepsilon^2}}$. Then when $n > N$, the above shows that $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$.

So by the def of the limit, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.