**Cauchy Integral Formula**

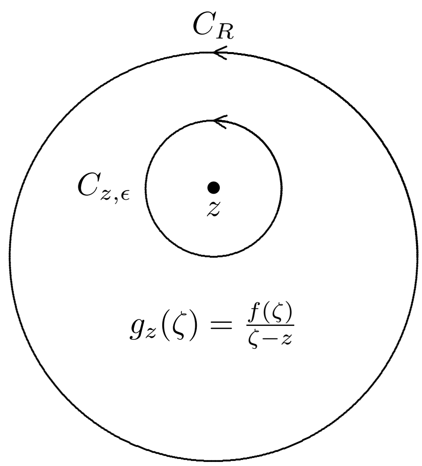
**Theorem.** *Let $f(z)$ be analytic in the closed region*

$$D_R = \{|z| \le R\}.$$

*Then for $|z| < R$,*

$$f(z) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

**Proof:** Fix $z$. For $\epsilon$ small, let $C_{z,\epsilon} = \{\zeta \,|\, |\zeta - z| = \epsilon\}$. Define $g_z(\zeta) = \dfrac{f(\zeta)}{\zeta - z}$. Then $g_z(\zeta)$ is an analytic function of $\zeta$ in the region between the two circles $C_{z,\epsilon}$ and $C_R$.




By the Two Circles Theorem

$$\oint_{C_R} g_z(\zeta)\, d\zeta = \oint_{C_{z,\epsilon}} g_z(\zeta)\, d\zeta,$$

$$\oint_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \oint_{C_{z,\epsilon}} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Since $f$ is analytic at $z$, for $\zeta$ near $z$,

$$f(\zeta) = f(z) + f'(z)(\zeta - z) + o(\zeta - z).$$

Thus

$$\begin{aligned}\oint_{C_{z,\epsilon}} \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \oint_{C_{z,\epsilon}} \frac{f(z) + f'(z)(\zeta - z) + o(\zeta - z)}{\zeta - z} d\zeta. \\ &= f(z) \oint_{C_{z,\epsilon}} \frac{1}{\zeta - z}\, d\zeta + f'(z) \oint_{C_{z,\epsilon}} 1\, d\zeta + \oint_{C_{z,\epsilon}} o(1)\, d\zeta \\ &= 2\pi i f(z) + 0 + o(\epsilon).\end{aligned}$$

Now let $\epsilon \to 0$ to obtain

$$\oint_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 2\pi i f(z).$$

**Power Series Representation of Analytic Functions**

We will show that the analytic function has a power series representation with a radius of convergence at least $R$.

Fix $\zeta$, $|\zeta| = R$, and fix $z$, $|z| < R$. Let $\theta = \left|\frac{z}{\zeta}\right|$. Then

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta\left(1 - \frac{z}{\zeta}\right)} \\ &= \frac{1}{\zeta}\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n \\ &= \frac{1}{\zeta}\sum_{n=0}^{N}\left(\frac{z}{\zeta}\right)^n + \frac{1}{\zeta}\frac{\left(\frac{z}{\zeta}\right)^{N+1}}{1 - \frac{z}{\zeta}}. \end{aligned}$$

It follows that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_R}\frac{f(\zeta)}{\zeta - z}\,d\zeta. \\ &= \frac{1}{2\pi i}\oint_{C_R} f(\zeta)\frac{1}{\zeta}\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n\,d\zeta \\ &= \frac{1}{2\pi i}\oint_{C_R} f(\zeta)\frac{1}{\zeta}\sum_{n=0}^{N}\left(\frac{z}{\zeta}\right)^n\,d\zeta \\ &+ \frac{1}{2\pi i}\oint_{C_R} f(\zeta)\frac{1}{\zeta}\frac{\left(\frac{z}{\zeta}\right)^{N+1}}{1 - \frac{z}{\zeta}}\,d\zeta \\ &= \sum_{n=0}^{N} a_n z^n + R_N(z), \end{aligned}$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_{C_R} f(\zeta)\frac{1}{\zeta^{n+1}}\,d\zeta, \\ R_N(z) &= \frac{1}{2\pi i}\oint_{C_R} f(\zeta)\frac{1}{\zeta}\frac{\left(\frac{z}{\zeta}\right)^{N+1}}{1 - \frac{z}{\zeta}}\,d\zeta, \\ |R_N(z)| &\le \max_{|\zeta|=R}|f(\zeta)|\,\frac{|\theta|^{N+1}}{1 - |\theta|}. \end{aligned}$$

**Theorem.** *Let $f(z)$ be analytic in the closed region*

$$D_R = \{|z| \le R\}.$$

*Then for $|z| < R$:*

- $$f(z) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

- $$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$
  $$a_n = \frac{1}{2\pi i} \oint_{C_R} f(\zeta) \frac{1}{\zeta^{n+1}}\, d\zeta,$$

- *For* $n = 0, 1, 2, \ldots,$ $f^{(n)}(z)$ *exists and*

$$\begin{aligned} f^{(n)}(z) &= \frac{n!}{2\pi i} \oint_{C_R} f(\zeta) \frac{1}{(\zeta - z)^{n+1}}\, d\zeta, \\ &= \frac{1}{2\pi i} \oint_{C_R} f(\zeta) \frac{d^n}{dz^n} \left\{ \frac{1}{\zeta - z} \right\} d\zeta. \end{aligned}$$