Differentiability and Derivatives

A real valued function of a real variable x is differentiable at x with derivative $f'(x)$ if

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + o\left(\Delta x\right).$$

We consider several types of functions:

Curves: Let $z(t) = x(t) + iy(t)$ be a path (a complex valued function of a real variable t). Then $z(t)$ is differentiable at t with derivative $z'(t)$ if

$$z(t+\Delta t) = z(t) + z'(t)\Delta t + o\left(\Delta t\right).$$

In this case $z'(t) = \frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$. Moreover, the derivative can be calculated as the limit of difference quotients:

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}.$$

Complex functions of a complex variable z: Let $f(z)$ be defined in an open set. Then f is differentiable at z with derivative $f'(z)$ if

$$f(z+\Delta z) = f(x) + f'(z)\Delta z + o\left(\Delta z\right).$$

The derivative can be calculated as the limit of difference quotients:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

While the formal definitions are the same, the requirements are quite different. If the variable, e.g., x, is real, there are only two ways that $\Delta x \to 0$ – from the right ($\Delta x > 0$) or from the left ($\Delta x > 0$). In the complex variable case, $|\Delta z| \to 0$, but Δz is allowed to have random directions as $\Delta z \to 0$. The existence of a limit $\lim_{\text{complex } z \to \cdot}$ is stronger than the concept $\lim_{\text{real } x \to \cdot}$.

If f is a differentiable function of the complex variable z in an open set or region, $f(z)$ is also called an *analytic* or *holomorphic* function[1].

Real [Complex] Functions of Two Variables (x, y): There is another concept of differentiability of functions of two (or more) variables (x, y). For simplicity write $P = (x, y)$ and $\Delta P = (\Delta x, \Delta y)$. Then a real (or complex) valued function G is differentiable at P if

$$G(P + \Delta P) = G(P) + \text{linear function of } \Delta P + o\left(\Delta P\right).$$

[1] See Knopp, **Elements of the Theory of Functions**, p. 101, and Knopp, **Theory of Functions, Part I**, p. 27. Thus if $f(z)$ is analytic at a point z_0, $f(z)$ is actually analytic in a neighborhood of z_0.

All linear functions of $\Delta P = (\Delta x, \Delta y)$ are of the form $a \cdot \Delta x + b \cdot \Delta y$. The numbers a and b can be calculated as the partial derivatives of G:

$$a = \frac{\partial G}{\partial x} = \lim_{\Delta x \to 0} \frac{G(x+\Delta x, y) - G(x,y)}{\Delta x},$$
$$b = \frac{\partial G}{\partial y} = \lim_{\Delta y \to 0} \frac{G(x, y+\Delta y) - G(x,y)}{\Delta y}.$$

The **derivative** of G at P is then the (complex) pair $< \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} >$ so that

$$G(P + \Delta P) = G(P) + \frac{\partial G}{\partial x} \cdot \Delta x + \frac{\partial G}{\partial y} \cdot \Delta y + o\left(|\Delta P|\right).$$

The [complex] pair $< \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} >$ is called the [*complex*] *gradient* of G and is written $\operatorname{grad} G$ or ∇G.

The formal expression $\frac{\partial G}{\partial x}\, dx + \frac{\partial G}{\partial y}\, dy$ is also called the differential dG of the function $G(x,y)$. We write

$$dG = \frac{\partial G}{\partial x}\, dx + \frac{\partial G}{\partial y}\, dy.$$

Back to Analytic (Holomorphic[2]) Functions: That a differentiable function G of two real variables $P = (x,y)$ is a *differentiable function of the complex variable* z or *analytic* (*holomorphic*) reduces to the statement:

If $P = (x,y)$, $z = x + iy$, there is a complex number $G'(z)$ such that

$$G'(z) \cdot (\Delta x + i\Delta y) = \frac{\partial G}{\partial x} \cdot \Delta x + \frac{\partial G}{\partial y} \cdot \Delta y.$$

The complex number $G'(z)$ can be calculated several ways:

- Let $\Delta y = 0$ and $\Delta x \neq 0$:
$$G'(z) = \frac{\partial G}{\partial x}$$
- Let $\Delta y \neq 0$ and $\Delta x = 0$:
$$iG'(z) = \frac{\partial G}{\partial y}, \text{ or } G'(z) = -i\frac{\partial G}{\partial y}$$
- $$G'(z) = \frac{\partial G}{\partial z} \overset{\text{def}}{\equiv} \frac{1}{2}\left(\frac{\partial G}{\partial x} - i\frac{\partial G}{\partial y}\right)$$

The formal expression $\frac{\partial G}{\partial x}\, dx + \frac{\partial G}{\partial y}\, dy$ is also called the differential dG of the function $G(x,y)$. If G is a differentiable function of the complex variable z, we can write formally:

$$dG = \frac{\partial G}{\partial x}\, dx + \frac{\partial G}{\partial y}\, dy = \frac{dG}{dz}\, dz = \frac{dG}{dz}\, dz.$$

[2] **The words *holomorphic* and *analytic* are used interchangably.**

The precise meaning of the statement $dG = \frac{dG}{dz}\,dz$ is that

$$G(z+\Delta z) - G(z) = \frac{dG}{dz} \cdot \Delta z + o\left(\Delta z\right).$$

Remarks on Analyticity and Partial Differential Equations (PDE)
If $G(x,y)$ is real differentiable,

$$\begin{aligned} dG &= \frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy, \\ dx &= \frac{1}{2}\left(dz + d\bar{z}\right), \\ dy &= \frac{1}{2i}\left(dz - d\bar{z}\right), \end{aligned}$$

so that

$$\begin{aligned} dG &= \frac{1}{2}\left(\frac{\partial G}{\partial x} - i\frac{\partial G}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial G}{\partial x} + i\frac{\partial G}{\partial y}\right)d\bar{z} \\ &= \frac{\partial G}{\partial z}\,dz + \frac{\partial G}{\partial \bar{z}}\,d\bar{z}. \end{aligned}$$

Thus the *analytic* functions are the real differentiable functions $G(x,y)$ which satisfy the partial differential equation

$$\frac{\partial G}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial G}{\partial x} + i\frac{\partial G}{\partial y}\right) = 0.$$