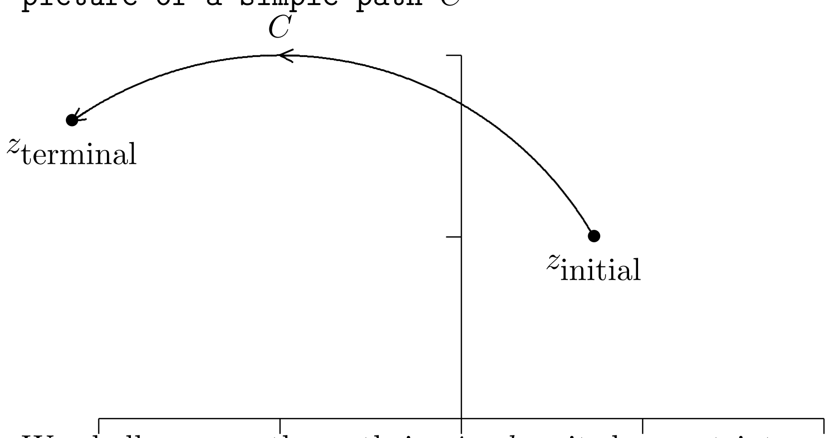
Paths and Integrals

A C^1 *path* C is a complex valued function

$$C : z = z(t) = x(t) + iy(t), a \leq t \leq b,$$

where $z(t)$ is continuously differentiable. The path C is represented by its image with an arrow drawn in the direction of increasing t.

`picture of a simple path` C



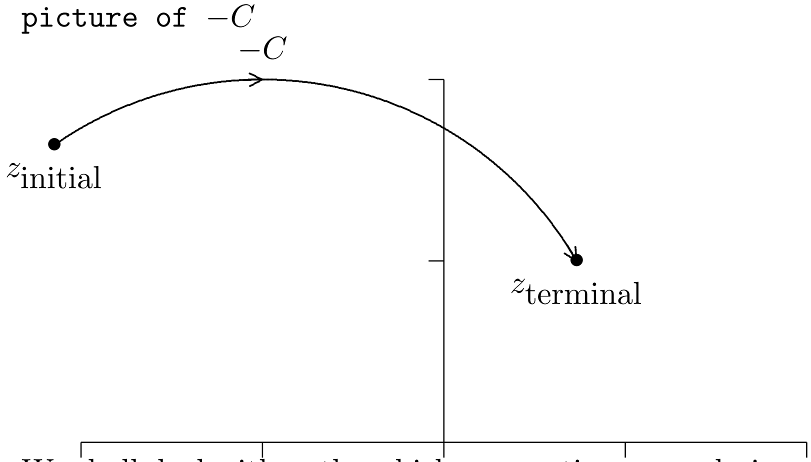
We shall assume the path is *simple* - it does not intersect itself except possibly at its endpoints.

The path is *closed* if it is simple and the endpoints are the same: $z(a) = z(b)$.

If C is a path, the path $-C$ is the path with the same image but traced in the opposite direction. If C is parameterized by $z_C(t), 0 \leq t \leq 1$, then $-C$ may be parameterized by

$$-C : z = z_{-C}(t) = z_C(1-t), 0 \leq t \leq 1.$$

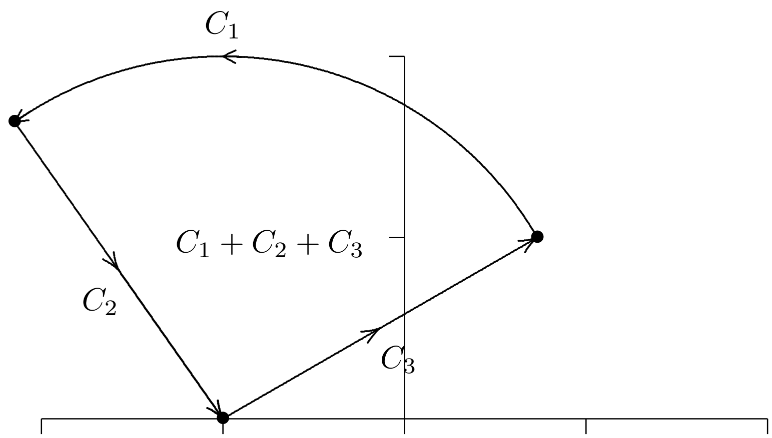
`picture of` $-C$



We shall deal with paths which are continuous and piecewise C^1. Such paths can written as a formal sum $C_1 + C_2 + \ldots + C_N$, where the terminal point of C_j is the initial point of C_{j+1}, $j = 1, \ldots, N-1$. [1]

[1] More generally, we can consider a *chain*: $C_1 + C_2 + \ldots + C_N$, a formal sum even when the components do not connect. The concept of *chain* is used in differential geometry.

picture of 3 arcs



For our purposes C will consist of a [small] number of arcs and line segments.

Integrals on Paths

Let C be a continuous and [piecewise] C^1 path and let $f(z)$ be a continuous function defined on C. Let C be parameterized by

$$C : z = z(t) = x(t) + iy(t), a \leq t \leq b.$$

Then the integral of $f(z)\,dz$ on C is defined as:

- The Quick Definition

$$\int_C f(z)\,dz = \int_a^b f(z(t))\frac{dz}{dt}\,dt,$$

where C is parameterized by

$$C : z = z(t) = x(t) + iy(t), a \leq t \leq b.$$

- The Riemann Sum Definition:

Let $\Pi : a = t_0 < t_1 < \ldots < t_M = b$, be a partition of $[a, b]$, and $z_j' = z(t_j')$ be a typical point in the image of $[t_j, t_{j+1}]$; define the *Riemann sum*

$$\begin{aligned} R(f, \Pi, z_j') &= \sum_{j=0}^{M-1} f(z_j')\,(z_{j+1} - z_j) \\ &\approx \sum_{j=0}^{M-1} f(z_j') \cdot z'(t_j') \cdot (t_{j+1} - t_j) \\ &= \sum_{z \text{along} C} f(z)\,\Delta z. \end{aligned}$$

Then

$$\begin{aligned} \int_C f(z)\,dz &= \lim_{\max|\Delta z| \to 0} R(f, \Pi, z_j') \\ &= \lim_{\max|\Delta z| \to 0} \sum_{z \text{along} C} f(z)\,\Delta z. \end{aligned}$$

The "quick" definition gives an effective way to compute $\int_C f(z)\,dz$. The "Riemann sum" definition emphasizes that $\int_C f(z)\,dz$ is independent of the parameterization of C. Either definition gives that

$$\int_C f(z) \pm g(z)\,dz = \int_C f(z)\,dz \pm \int_C g(z)\,dz,$$
$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz,$$
$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

The last relation says that for fixed $f(z)$, $\int_C f(z)\,dz$ is additive as a map on sums of paths or chains.[2]

A Version of the Fundamental Theorem of Calculus

Theorem (FTC Version I). *Let C be a continuous piecewise C^1 path and let $F(z)$ be analytic at every point on C. Then*

$$\int_C F'(z)\,dz = F(z(b)) - F(z(a)).$$

where

$$C : z = z(t) = x(t) + iy(t), a \le t \le b.$$

is a parameterization of C.

Proof: By the chain rule for differentiation

$$\frac{d}{dt}F(z(t)) = F'(z(t))\frac{dz}{dt},$$

so by FTC (Version I) for functions of a real variable

$$\begin{aligned}\int_C F'(z)\,dz &= \int_a^b \frac{d}{dt}F(z(t))\,dt \\ &= F(z(b)) - F(z(a)).\end{aligned}$$

The Most Important Path Integral

If the curve C is simple and closed and traversed in the *counterclockwise* direction, we often write

$$\int_C f(z)\,dz = \oint_C f(z)\,dz$$

The most important integral is the integral of $f(z) = \frac{1}{z}$ around a circle containing the origin.

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2 Compare to the calculus result

$$\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt.$$

Theorem. *Let C_R be the circle of radius R, centered at 0, and traversed in the counterclockwise direction. Then*

$$\oint_{C_R} \frac{1}{z}\, dz = 2\pi i.$$

Proof: C_R can be parameterized by the angle t, $0 \le t \le 2\pi$:

$$\begin{aligned} C_R : z(t) &= Re^{it} \\ &= R(\cos(t) + i\sin(t)), \\ dz &= iRe^{it}\, dt. \end{aligned}$$

Then

$$\begin{aligned} \oint_{C_R} \frac{1}{z}\, dz &= \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}}\, dt \\ &= \int_0^{2\pi} i\, dt \\ &= 2\pi i. \end{aligned}$$

Exercise

Use the above Fundamental Theorem Calculus or the parametric representation to show that for n an integer, $n \neq -1$,

$$\oint_{C_R} z^n\, dz = 0.$$