Removable and Nonremovable Singularities

Let Ω be an open set. Let $f(z)$ be analytic in Ω. A complex number z_0 in the boundary of Ω is a *removable singularity* for f [with respect to Ω] if there is a neighborhood[1] $B_{z_0,\epsilon}$ of z_0, and a function $g(z)$, analytic in $B_{z_0,\epsilon}$, such that $g(z) = f(z)$ in $B_{z_0,\epsilon} \cap \Omega$.

In this case the function $f(z)$ can be *continued* analytically to a larger open set, $\Omega \cup B_{z_0,\epsilon}$. The process is called *analytic continuation.*

Examples

- For example, if $\Omega = B_{0,r} \setminus \{0\}$, and $f(z)$ is analytic and bounded in Ω, then the point $z = 0$ is a removable singularity for $f(z)$.
- if $\Omega = B_{0,r}$, and $z_0 \in \partial\Omega$ and $f(z)$ is analytic in Ω, but $f(z)$ is unbounded for z near z_0 then the point $z = z_0$ is a *nonremovable* singularity for $f(z)$ with respect to Ω.
- The function $f(z) = \ln(z)$ may be defined in the set
$$\Omega = \{z = x + iy \mid x > 0\}$$
as
$$\ln(z) = \ln|z| + i\arg(z), -\frac{\pi}{2} < \arg(z) < \frac{\pi}{2}.$$
The only point on the imaginary axis, $\partial\Omega$, which is not a removable singularity for $f(z)$ with respect to Ω is $z = 0$.
- The function $f_1(z) = \ln(z)$ may be defined in the set
$$\Omega_1 = \mathbf{C} \setminus \{z = x + i0 \mid x \leq 0\}$$
as
$$\ln(z) = \ln|z| + i\arg(z), -\pi < \arg(z) < \pi.$$
Then all points on the nonnegative real axis, $\partial\Omega_1$, are nonremovable singularities for $f_1(z)$ with respect to Ω_1. This example shows that whether the singularity is removable depends on the original domain of definition.

Radius of Convergence for Power Series

Suppose that $f(z)$ is analytic in $|z| < R$ and there is a point z_0, $|z_0| = R$, such that z_0 is a nonremovable singularity for $f(z)$ with respect to $B_{0,R}$. Then the radius convergence for the series
$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$
is exactly R.

Proof: We know that the radius of convergence is at least R. If the series converges for $|z| < R_1$, $R_1 > R$, let
$$g(z) = \sum_{n=0}^{\infty} a_n z^n, |z| < R_1.$$
Then $g(z) = f(z)$, $|z| < R_1$, but $g(z)$ is analytic at $z = z_0$.

[1] We use the notation
$$B_{z_0,\epsilon} = \{z \mid |z - z_0| < \epsilon\}.$$
for the open disk centered at z_0 of radius ϵ.