Derivatives of Log & Exp functions

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}e^x = e^x$$

CHAIN RULE

$$\frac{d}{dx}\ln(u(x)) = \frac{1}{u(x)} = \frac{\frac{du}{dx}}{u}$$

[= relative rate of change of u!

[N.B. $u > 0$]

$$\frac{d}{dx}e^{u(x)} = e^{u(x)}\frac{du}{dx}$$

SPECIAL CASE(S)

$$\frac{d}{dx}\ln(ax) = \frac{1}{x} \quad ?$$

$$\frac{d}{dx}e^{bx} = be^{bx}$$

$$\frac{d}{dx}e^{kx} = ke^{kx}$$

$$\frac{d}{dx}e^{-kx} = -ke^{-kx}$$

If u were negative

$$\frac{d}{dx}\ln(|u|) = \frac{1}{|u|}\,\text{Sign}\,u(x)$$

$$= \frac{1}{u}$$

PATTERNS ln...

$$x^k \to kx^{k-1}$$

[$k=0$ ~~ln~~ $x^0 \to 0$

Missing is a fn whose derivative is x^{-1}. "$\ln(|x|)$" fills the gap

More useful are the "logarithm rules"

$$\cancel{\log}\ \ln(u \cdot v) = \ln u + \ln(v)$$

$$\to \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx}$$

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Silly example 4.3.2

$$f(x) = \frac{\ln \sqrt[3]{x^2}}{x^4} \quad \text{QUOTIENT}$$

$$= \frac{2}{3} \frac{\ln(x)}{x^4}$$

$$\longrightarrow \frac{2}{3} \frac{x^4 \cdot \frac{1}{x} - \ln(x) \cdot 4x^3}{x^8}$$

$$= \frac{2}{3} \frac{x^3 [1 - 4\ln(x)]}{x^8}$$

$$= \frac{2}{3} \frac{[1 - 4\ln(x)]}{x^5}$$, certainly $x > 0$.

Less silly 4.3.4 $\frac{d}{dx}(\ln(2x^3+1))$

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price-demand and revenue, profit, etc.

Exponential Exercise:

$$y = \frac{e^{-3x}}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)(-3e^{-3x}) - e^{-3x}(2x)}{(x^2+1)^2}$$

cut mo $$\frac{e^{-3x}(-3(x^2+1) - 2x)}{(x^2+1)^2}$$

$\boxed{NB}$ "e^{-3x}" is common factor

$$= \frac{e^{-3x}}{(\quad)^2} \Big| -3x^2 - 3 - 2x$$

$3x^2 + 2x + 3$

$= (x \quad 3)(3x \quad 1)$

NO INTEGER ROOTS

Roots $x = \frac{-2 \pm \sqrt{4 - 36}}{2 \cdot 3}$

No real roots!

~~p_{n}~~ REVENUE Example 4.3.11

Suppose that price and demand are related by

$$q = D(p) = 5000\, e^{-.02p}$$

~~Find the maximum revenue~~ Find the price for MAX Reven

$$R(p) = p \cdot q = 5000\, p\, e^{-.02p}$$

$$\frac{dR}{dp} = 5000 \left[p \cdot e^{-.02p}(-.02) + e^{-.02p} \right]$$

Solve $=0$. SIMPLIFICATION AVAILABLE

$$\frac{dR}{dp} = 5000 \underbrace{e^{-.02p}}_{\text{always } \oplus \text{ never } 0} (-.02p + 1)$$

Crit. no $p = \frac{1}{.02} = 50$ (hand or calculator)

ACTUAL MAX NOTE FORM

$$5000\, e^{-.02p}(-.02p + 1)$$

$+$ $\nearrow$ $\searrow$ $\ominus$ Rel. MAX

0 50

Price Elasticity of demand $\boxed{E(p) = \frac{p}{q} \frac{dq}{dp}}$

$$\frac{dq}{dp} = 5000(-.02)\, e^{-.02p}$$

$$E(p) = \frac{p}{\cancel{5000(\ )\, e^{-.02p}}}\ \cancel{5000(-.02)\, e^{-.02p}}$$

MAGIC! $E(p) = -.02p$. Unit Elasticity when $-.02p = -1$, or $p = 50$

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Logarithmic Differentiation

Treat combinations of $\left[\begin{array}{l}\text{prod} \\ \text{quot} \\ \text{powers} \\ \text{roots}\end{array}\right]$ not so good on sums, differences

Example 4.3.12

Diff $y = \dfrac{\sqrt[3]{x+1}}{(1-3x)^4}$ possible, understand the formula

$$\ln(y) = \ln\left(\frac{\sqrt[3]{x+1}}{(1-3x)^4}\right)$$

$$= \frac{1}{3}\ln(x+1) - 4\ln(1-3x)$$

$$\frac{d}{dx} \cdot \quad = \frac{1}{3}\,\frac{1}{x+1} - 4\cdot\frac{1}{1-3x}(-3)$$

$$= \frac{1}{3}\cdot\frac{1}{x+1} + \frac{12}{(1-3x)}$$

N.B. $1-3x \neq 0$ actually $1-3x > 0$

$x \neq -1$, $x+1$ ~~pos~~

$y = b^x$ b fixed base > 0, $b \neq 1$.

1° LOGDIFF $\ln(y) = \ln(b^x) = x\ln(b)$

Chain $\dfrac{\frac{dy}{dx}}{y} = \ln(b)$

$\dfrac{dy}{dx} = y\ln(b) = b^x \ln(b)$

2° "Definition" of $b^x = e^{x\ln(b)}$

$$\frac{d}{dx} := e^{x\ln(b)} \cdot \ln(b) = b^x \ln(b)$$

Relative Rate of change of u $\quad \dfrac{\frac{du}{dx}}{u} = 100$ times percentage rate of change

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$\boxed{4.3.14}$ Export 3 goods Wheat Steel Oil

~~With~~ $W \approx 4$, $S(t_0) = 7$ $\quad O(t_0) = 10$ (billion

Growing at [-3%] $\quad +8\%$ $\quad +13\%$ 5/

$$\frac{4}{21}(-.03) + \frac{7}{21}.08 + \frac{10}{21}(.13) =$$

$$\frac{\frac{dW}{dt}}{W} \approx -.13 \qquad \frac{dS}{dt} \approx .08S \qquad \frac{dO}{dt} \approx .13O$$

$$\cancel{\frac{dW}{dt}} \rightarrow \frac{\frac{d}{dt}(W+S+O)}{W+S+O}$$

$$\approx \frac{-.13\overset{4}{W} + .08\overset{7}{S} + .13\overset{10}{O}}{\underset{4+7+10}{W+S+O}}$$

$$= -.13\frac{4}{21} + .08\frac{7}{21} + .13\frac{10}{21}$$