University of Illinois at Chicago

MATH165 - Calculus for Business (Spring 2008)

Exam 1 (2008-02-15)

Solution

Question 1. Find $f(-3)$ where $f(t) = \begin{cases} -2t+2 & \text{if } t < 1, \\ t^2+3 & \text{if } t \geq 1 \end{cases}$.

Solution to Question 1. Note that $-3 < -1$, so $f(t)$ takes the expression $-2t+2$ when $t = -3$. Hence $f(-3) = -2(-3) + 2 = \boxed{8}$.

Question 2. If $f(x) = \frac{1}{x}$, simplify $\frac{f(x+h) - f(x)}{h}$.

Solution to Question 2.

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} = \frac{1}{h} \cdot \frac{x-(x+h)}{x(x+h)} = \frac{1}{h} \cdot \frac{-h}{x(x+h)} = \boxed{\frac{-1}{x(x+h)}}$$

Question 3. Differentiate: $f(x) = \frac{x^2}{x-2}$.

Solution to Question 3. Recall Quotient Rule: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{x}}{v^2}$.
Let $u(x) = x^2$ and $v(x) = x - 2$. We have $f(x) = \frac{u(x)}{v(x)}$, $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$. Therefore,

$$f'(x) = \frac{(x-2)(2x) - x^2(1)}{(x-2)^2} = \frac{2x^2 - 4x - x^2}{(x-2)^2} = \boxed{\frac{x^2-4x}{(x-2)^2}}.$$

Question 4. Find the limit: $\lim_{x\to 12} \frac{x^2-144}{x^2-12}$.

Solution to Question 4.

$$\lim_{x\to 12} \frac{x^2-144}{x-12} = \lim_{x\to 12} \frac{(x-12)(x+12)}{x-12} = \lim_{x\to 12}(x+12) = 12 + 12 = \boxed{24}$$

Question 5. What is the rate of change of $f(t) = \frac{2t-5}{t+6}$ with respect to t when $t = 11$?

Solution to Question 5. It suffices to find the value of $f'(11)$. Let $u = 2t - 5$ and $v = t + 5$. Then $f(t) = \frac{u(t)}{v(t)}$, $u'(t) = 2$ and $v'(t) = 1$. By Quotient Rule (see Question 3), we have

$$f'(t) = \frac{(t+6)(2) - (2t-5)(1)}{(t+6)^2} = \frac{2t + 12 - 2t + 5}{(t+6)^2} = \frac{17}{(t+6)^2}.$$

Therefore, $f'(11) = \frac{17}{(11+6)^2} = \frac{17}{289} = \boxed{\frac{1}{17}}$.

Question 6. Find $\frac{dy}{dx}$, where $xy^3 - 3x^2 = 7y$.

Solution to Question 6. We differentiate the equation with respect to x to obtain the answer.

$$\begin{aligned}
\frac{d(xy^3 - 3x^2)}{dx} &= \frac{d(7y)}{dx} \\
\frac{d(xy^3)}{dx} - \frac{d(3x^2)}{dx} &= 7\frac{dy}{dx} && \left(\text{Sum rule: } \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}\right) \\
x\frac{dy^3}{dx} + y^3\frac{dx}{dx} - 6x &= 7\frac{dy}{dx} && \left(\text{Product rule: } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right) \\
x\frac{dy^3}{dy}\frac{dy}{dx} + y^3 \cdot 1 - 6x &= 7\frac{dy}{dx} && \left(\text{Chain rule: } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}\right) \\
x(3y^2)\frac{dy}{dx} + y^3 - 6x &= 7\frac{dy}{dx} \\
(3xy^2 - 7)\frac{dy}{dx} &= 6x - y^3 \\
\frac{dy}{dx} &= \boxed{\frac{6x - y^3}{3xy^2 - 7}}
\end{aligned}$$

Question 7. The derivative of a function is $f'(x) = x^3 + x - 1$. Estimate the root of $f'(x) = 0$. *Show work* means *a brief explanation of how you used your calculator.*

Solution to Question 7. There are many ways to write the solutions. Here is one of the ways:

1. Plot the graph $g(x) = x^3 + x - 1$ [with the command `x^3 + x - 1`].
2. Magnify the portion of the graph for $0.5 \le x \le 0.7$ [by setting `xmin = 0.5` and `xmax = 0.7` for the window of the graph].
3. Locate the intersection of the curve and y-axis. The x-coordinate of the intersection is about 0.68. Therefore, the root is about $\boxed{x = 0.68}$.

Question 8. An equation for the tangent line to the curve $y = (x^2 + x - 1)^7$ at the point where $x = 1$ is: ...

Solution to Question 8. Note that the slope of the tangent is $\left.\frac{dy}{dx}\right|_{x=1}$ and the tangent passes through $(1, (1^2 + 1 - 1)^7) = (1, 1)$. From chain rule (See Question 6), we have

$$\frac{dy}{dx} = \frac{d(x^2 + x - 1)^7}{dx} = \frac{d(x^2 + x - 1)^7}{d(x^2 + x - 1)} \cdot \frac{d(x^2 + x - 1)}{dx} = 7(x^2 + x - 1)^6(2x + 1).$$

By substituting $x = 1$, we get $\left.\frac{dy}{dx}\right|_{x=1} = 7(1 + 1 - 1)^6[2(1) + 1] = 21$. Recall that the equation of a line with slope m which passes through (x_0, y_0) can be obtained from its *point-slope form* $y - y_0 = m(x - x_0)$. Therefore, the equation of the tangent is $y - 1 = 21(x - 1)$, i.e. $\boxed{y = 21x - 20}$.

Question 9. A manufacturer has determined that q units are sold when the price, p, is given by $p = 120 - 2q$. The marginal revenue $\frac{\mathrm{d}R}{\mathrm{d}q}$ is ... for $q = 20$ and ... for $q = 40$.

Solution to Question 9. Note that $R = pq = q(120 - 2q)$. We obtain $\frac{\mathrm{d}R}{\mathrm{d}q} = \frac{\mathrm{d}(q(120-2q))}{\mathrm{d}q} = \frac{\mathrm{d}(120q - 2q^2)}{\mathrm{d}q} = 120 - 4q$. Therefore, $\left.\frac{\mathrm{d}R}{\mathrm{d}q}\right|_{q=20} = 120 - 4(20) = 40$ and $\left.\frac{\mathrm{d}R}{\mathrm{d}q}\right|_{q=40} = 120 - 4(40) = -40$. Hence, $\frac{\mathrm{d}R}{\mathrm{d}q}$ is $\boxed{\text{positive}}$ for $q = 20$ and $\boxed{\text{negative}}$ for $q = 40$.

Question 10. An efficiency study of the morning shift at a certain factory indicates that an average worker arriving on the job at 8:00 A.M. will have assembled $Q(t) = -t^3 + 9t^2 - 2t$ transistor radios t hours later. Approximately how many radios will the worker assemble between 10:00 and 10:30 A.M.?

Solution to Question 10. Note that 10:00 and 10:30 A.M. are 2 and 2.5 hours after 8:00 A.M. respectively. From the function $Q(t)$, we know the number of transistor radios produced before 10:00 and 10:30 A.M. will be $Q(2) = -2^3 + 9\times 2^2 - 2\times 2 = 24$ and $Q(2.5) = -2.5^3 + 9\times 2.5^2 - 2\times 2.5 = 35.625$. Therefore, about $35.625 - 24 = 11.625 \approx \boxed{11}$ radios are assembled between 10:00 and 10:30 A.M.

Remark. You may also approximate $Q(2.5) - Q(2) \approx (2.5-2)Q'(2) = 0.5[-3(2)^2 + 18(2) - 2] = \boxed{11}$. Yet, the approximation is not very good as 2.5 and 2 are not 'close enough'.

Question 11. At a certain factory, the total cost of manufacturing q units during the daily production run is $C(q) = 0.3q^2 + 0.7q + 800$ dollars. It has been determined that approximately $t^2 + 70t$ units are manufactured during the first t hours of a product run. Compute the rate at which the total manufacturing cost C is changing **with respect to time** 2 hours after production begins.

Solution to Question 11. Note that $q(t) = t^2 + 70t$. The required rate $\frac{\mathrm{d}C}{\mathrm{d}t}$ can be computed by chain rule (See Question 6):

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\mathrm{d}C}{\mathrm{d}q}\cdot\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}(0.3q^2+0.7q+800)}{\mathrm{d}q}\cdot\frac{\mathrm{d}(t^2+70t)}{\mathrm{d}t} &= (0.6q+0.7)(2t+70)\\ &= [0.6(t^2+70t)+0.7](2t+70)\\ &= (0.6t^2+42t+0.7)(2t+70)\\ &= 1.2t^3+126t^2+2941.4t+49\end{aligned}$$

Therefore, C is increasing at a rate of $\left.\frac{\mathrm{d}C}{\mathrm{d}t}\right|_{t=2} = 1.2(2)^3 + 126(2)^2 + 2941.4(2) + 49 = \boxed{6445.4}$ dollars per hour at the moment 2 hours after production begins.

Question 12. Find all points on the graph of the function $f(x) = \dfrac{x^2}{x+2}$ where the tangent line is horizontal.

Solution to Question 12. It suffices to solve the equation $f'(x) = 0$. From Quotient Rule (See Question 3), we have $f'(x) = \dfrac{(x+2)(2x) - x^2(1)}{(x+2)^2} = \dfrac{2x^2+4x-x^2}{(x+2)^2} = \dfrac{x^2+4x}{(x+2)^2}$. We solve the equation $f'(x) = 0$ as follows:

$$\begin{aligned} f'(x) &= 0 \\ \frac{x^2+4x}{(x+2)^2} &= 0 \\ (x+4)x &= 0 \\ x = -4 &\text{ or } x = 0. \end{aligned}$$

Therefore, the points $(-4, f(-4)) = \left(-4, \dfrac{(-4)^2}{-4+2}\right) = \boxed{(-4,-8)}$ and $(0, f(0)) = \left(0, \dfrac{0^2}{0+2}\right) = \boxed{(0,0)}$ are the points where the tangent line is horizontal.

Question 13. (LINEAR PRICE DEMAND) A commuter transit line now transports about 6000 riders per day. The current fare is \$3.00 per rider. Every fare increase of \$0.10 will result in a loss of approximately 200 riders per day. Express the daily number of riders, q, as a function of the fare, p (in dollars).

Solution to Question 13. Note that the rate of decrease in riders per day per dollar increment in fare is 2000 riders per day per dollar. If the price rises from 3 dollars to p dollars, the increment is $(p-3)$ dollars. It will lead to a loss of $2000(p-3)$ riders per day. Therefore, $\boxed{q = 6000 - 2000(p-3)}$.

Extra Credit (following Question 13). For what value of p is the total revenue maximized?

Solution to Extra Credit Problem. Note that the total revenue R (in dollar) is the product of p and q, i.e. $R = p[6000 - 2000(p-3)] = 12000p - 2000p^2$. To find the maximum value of R, we may solve $\dfrac{\mathrm{d}R}{\mathrm{d}p} = 0$ as follows:

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}p} &= 0 \\ \frac{\mathrm{d}(12000p - 2000p^2)}{\mathrm{d}p} &= 0 \\ 12000 - 4000p &= 0 \\ p &= 3. \end{aligned}$$

Therefore, the total revenue is maximized when $\boxed{p = 3}$. (One may check with the second derivative that it is a maximum.)