Convergence and Absolute Convergence

- $\displaystyle\sum_{*}^{\infty} C_N$ CONVerges iff
$$\lim_{N\to\infty}\sum_{*}^{N} C_n$$
exists (finite).
Compare to
- $\displaystyle\int_{*}^{\infty} f(x)\,dx$ CONVerges iff
$$\lim_{X\to\infty}\int_{*}^{X} f(x)\,dx$$
exists (finite).
- $\displaystyle\sum_{*}^{\infty} C_n$ CONVerges ABSsolutely iff $\displaystyle\sum_{*}^{\infty} |C_n|$ converges or
$$\lim_{N\to\infty}\sum_{*}^{N} |C_n|$$
exists (finite).
Compare to
- $\displaystyle\int_{*}^{\infty} f(x)\,dx$ CONVerges ABSsolutely iff $\displaystyle\int_{*}^{\infty} |f(x)|\,dx$ converges or
$$\lim_{X\to\infty}\int_{*}^{X} |f(x)|\,dx$$
exists (finite).
- $\displaystyle\sum_{*}^{\infty} |C_n|$ converges iff the sequence $\displaystyle\sum_{*}^{N} |C_n|$ is bounded.
- $\displaystyle\int_{*}^{\infty} |f(x)|$ converges iff the function $\displaystyle\int_{*}^{X} |f(x)|\,dx$ is bounded (as $X\to\infty$).

Comparison Test for Absolute Convergence

- If
$$0 \le A_n \le B_n,$$
then
$$0 \le \sum_{*}^{\infty} A_n \le \sum_{*}^{\infty} B_n.$$

So that if the bigger series $\sum_{*}^{\infty} B_n$ CONVerges, the smaller series $\sum_{*}^{\infty} A_n$ CONVerges also.

If the smaller series $\sum_{*}^{\infty} A_n$ DIVerges, the bigger series $\sum_{*}^{\infty} A_n$ DIVerges also.

- If

$$0 \leq f(x) \leq g(x),$$

then

$$0 \leq \int_{*}^{\infty} f(x)\,dx \leq \int_{*}^{\infty} g(x)\,dx.$$

So that if the bigger integral $\int_{*}^{\infty} g(x)\,dx$ CONVerges, the smaller integral $\int_{*}^{\infty} f(x)\,dx$ CONVerges also.

If the smaller integral $\int_{*}^{\infty} f(x)\,dx$ DIVerges, the bigger integral $\int_{*}^{\infty} g(x)\,dx$ DIVerges also.

Ratio Test for ABSolute CONVergence

For the series $\sum_{*}^{\infty} C_N$, suppose that

$$\lim_{n\to\infty} \left| \frac{C_{n+1}}{C_n} \right| = L.$$

- If $0 \leq L < 1$, the series $\sum_{*}^{\infty} C_N$ CONVerges ABSolutely.
- If $1 < L \leq \infty$, the series $\sum_{*}^{\infty} C_N$ DIVerges.
- If $L = 1$, we are not sure – additional information is needed about DIVergence or CONVergence and/or ABS0lute CONVergence.

Power Series, Radius of Convergence, and Interval of Convergence

- For a power series $\sum_{n=0}^{\infty} a_n x^n$, there is a number R, $0 \leq R \leq \infty$ for which

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{CONVerges ABSolutely for } |x| < R, \\ \text{DIVerges for } |x| > R. \end{cases}$$

The number R is called the *radius of convergence* of the power series. R can often be determined by the Ratio Test.

- If the power series $\sum_{n=0}^{\infty} a_n x^n$, converges for $x = x_0$, then for all x, $|x| < |x_0|$ the power series CONVerges ABSolutely. Thus the *radius of convergence*, R,is greater than or equal $|x_0|$.

- If $f(x)$ is represented by a convergent power series for $|x| < R$, then for $|x| < R$, its derivative is represented by the convergent series $\sum_{n=1}^{\infty} n a_n x^{n-1}$:

  If
$$f(x) = \sum_{n=0}^{\infty} a_n x^n, |x| < R,$$
then
$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, |x| < R,$$
and
$$\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n, |x| < R$$