MthT 430 Review 2006

Definitions

1. Define (ϵ–δ): $\lim_{x \to a} f(x) = L$.

2. Define: $\lim_{x \to a^-} f(x) = L$.

3. Define: The function f is continuous at a.

4. Define: The set of numbers A is bounded above.

5. Define: The number b is the *least upper bound* of a set of numbers A.

6. Define: The domain of a function f.

7. (See chap5cproj) Is the following "definition" correct? Explain your answer!

Definition HH.
$$\lim_{x \to a} f(x) = L$$
means: For any $\epsilon > 0$, *there is a* $\delta > 0$ *such that, for all* x, $|f(x) - L| < \epsilon$ *and* $0 < |x - a| < \delta$.

Examples

8. Give an example of two functions f and g such that $f \circ g = g \circ f$. Be sure to verify that the domains are the same.

9. Give an example of a bounded function f defined for all real numbers such that $\lim_{x \to 0} f(x)$ does not exist.

10. Give an example of a bounded set of numbers A which has a greatest element. Give the least upper bound of this set A.

11. Give an example of a nonempty bounded set of numbers A which has no greatest element. Give the least upper bound of this set A.

12. Give an example of a nonempty bounded set A_Q *of rational numbers* whose least upper bound *is not a rational number*.

13. Give an example of a function f with domain $(0, 1)$ such that

- f is continuous on $(0, 1)$ and
- f is not bounded above on $(0, 1)$.

14. Give an example of a function f with domain $[0, 1]$ such that

- f is continuous on $[0, 1]$ except at $x = \frac{1}{2}$ and
- f is not bounded above on $[0, 1]$.

15. Give an example of a function f with domain $[0, 1]$ such that

- f is continuous on $[0, 1]$ except at $x = \frac{1}{2}$ and
- f is bounded above on $[0, 1]$, and
- f does not assume a maximum value on $[0, 1]$.

16. Find the decimal and binary expansions of $x = \dfrac{1}{5}$.

17. Express as a rational number $x = \dfrac{p}{q}$, p, q natural numbers.

- $x = 0._{\text{bin}}\overline{01}$ (Binary or Base 2)
- $x = 0.\overline{01}$ (Decimal or Base 10)

Proofs

18. Using (P1 – P9), show that

$$-(a \cdot b) = (-a) \cdot b.$$

19. Using (P1 – P12), show that

$$a \le b$$

if and only if:

$$\text{For every } \epsilon > 0, a < b + \epsilon.$$

20. Prove by mathematical induction (PMI) or otherwise:

$$1^3 + \cdots + n^3 = (1 + \cdots + n)^2 .$$

21. Let f be defined on $[0,1)$ be such that

- f is increasing on $[0,1)$ (If $0 \leq x_1 < x_2 < 1$, then $f(x_1) < f(x_2)$.)
- f is bounded above on $[0,1)$.

Prove that

$$\lim_{x \to 1^-} f(x) = L$$

exists.

Hint: State precisely the version of (P13) that you use.

22. Prove $(\epsilon - \delta)$:

Theorem. *If*

$$\lim_{x \to a} f(x) = L \text{ and}$$
$$\lim_{x \to a} g(x) = M,$$

then

$$\lim_{x \to a} (f(x) + g(x)) = L + M.$$

23. Let A and B be nonempty bounded sets. Define

$$A + B = \{x | x = a + b, a \in A, b \in B\}$$

Show that

$$\inf (A + B) = \inf A + \inf B.$$

Qualitative Properties of Functions

24. Water drips very slowly into a circular bottle (beaker, flask) so that the graph of the Height (in cm) as a function of Volume (in cm^3) is shown below.

Construct your own (not too complicated) Height–Volume Graph!

Draw a side view of the bottle. Carefully explain as many features as you can about the shape of the bottle and explain how they are related to the Height–Volume graph.

Essay

25. (Letter Grade: A - E) In the exam booklet, write an essay on a topic of your choice that is very relevant to the material considered in the course. Your essay should include at least one substantial example and at least one substantial theorem and its proof.