MthT 430 2007 Midterm Assessment Remarks

I. Definitions

1. (10 points) Define (ϵ–δ): $\lim_{x \to a} f(x) = L$.

2. (10 points) Give an example that shows that the following definition of $\lim_{x \to a} f(x) = L$ is *not* correct:

   For every $\delta > 0$, there is an $\epsilon > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Let $f(x) = 1/x$, $a = 1$. Then $\lim_{x \to 1} f(x) = 1$, but the proposed definition is not satisfied for $\delta = 1$.

II. Examples

3. (10 points) Give an example of two functions f and g such that f/g and g/f have **different nonempty domains**. Be sure to specify domain (f), domain (g), domain (f/g), and domain (g/f).

4. (20 points) Let

$$F(x) = \sqrt{x^2 - 1},$$
$$G(x) = \sqrt{x^2 + 1}$$

Describe:

- domain (F) and domain (G).
- domain $(F + G)$
- domain $(G \circ F)$

We have that

$$\begin{aligned} (G \circ F)(x) &= \sqrt{\left(\sqrt{x^2 - 1}\right)^2 + 1} \\ &= \sqrt{(x^2 - 1) + 1} \\ &= \sqrt{x^2} \\ &= |x| . \end{aligned}$$

. The intermediate steps are defined iff $x^2 - 1 \geq 0$, so that domain $(F \circ G) = \{|x| \geq 1\}$.

- domain $(F \circ G)$
- domain $\left(\dfrac{F}{G}\right)$
- domain $\left(\dfrac{G}{F}\right)$

5. (10 points) For $a > 0$, find

$$\lim_{h \to 0} \frac{\sqrt{a + h} - \sqrt{a}}{h}.$$

6. (10 points) Give an example of two functions f, g, such that $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = 0$, and $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = 1$. Be sure to specify domain (f), domain (g), domain (f/g), and domain (g/f).

III. Proofs

7. (15 points Prove: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} (f+g)(x) = L + M$.

8. (15 points) Prove: If g is continuous at a, $g(a) \neq 0$, then there is a $\delta > 0$ for which $(a-\delta, a+\delta)$ is contained in the domain of $\dfrac{1}{g}$.

9. (15 points) Show, using only P1 – P9: For all numbers a, b,

$$-(a\cdot b) = (-a)\cdot b. \tag{$\clubsuit$}$$

You may abbreviate (*associative, distributive, trichotomy, ...*).

$$\begin{aligned} (ab) + (-a)\,b &= (a + (-a))\,b && (P9) \\ &= 0\cdot b && (P3) \\ &= 0 && \text{(in class)} \end{aligned}$$

The proof is finished. Optionally, add $-(ab)$ to both sides of the equation $(ab) + (-a)\,b = 0$.

Extra Credit: Show that for $a \neq 0$, $(-a)^{-1} = -\left(a^{-1}\right)$. **Hint:** Use ($\clubsuit$).

$$\begin{aligned} 1 &= -[-\left(aa^{-1}\right)] \\ &= -[a\left(-\left(a^{-1}\right)\right)] && (\clubsuit) \\ &= (-a)\left(-\left(a^{-1}\right)\right) && (\clubsuit) \end{aligned}$$

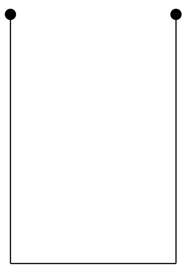
Therefore, $-\left(a^{-1}\right) = (-a)^{-1}$.

10. (15 points) Show by mathematical induction or otherwise: For all natural numbers $n = 1, 2, \ldots$,

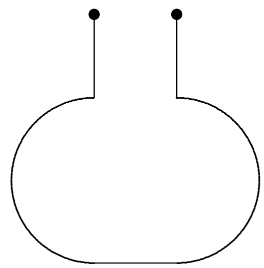
$$1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^2\,(n+1)^2\,.$$

IV. Qualitative Properties of Functions

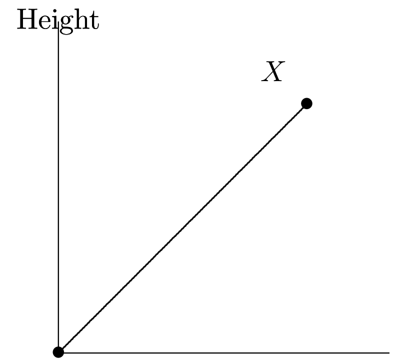
11. (20 points) The graph below shows how the height of a liquid in a Boiling Flask Z varies as water is steadily dripped into it. Copy the graph, and *on the same diagram* show the height–volume relationship for the Boiling Flask Z.



Beaker X



Boiling Flask Z



Describe the features of the graph you have drawn. Your description should include

- The domain of the function $[0, \text{Volume of Flask}]$
- The range of the function $[0, \text{Height of Flask}]$
- The intervals of monotonicity (Increasing, Decreasing)
- The intervals of constant concavity and/or linearity
- Other observations ...

A person reading your description of the graph should be able to reproduce the graph of the function (and if she's good, guess that it came from something shaped like a Boiling Flask).

N.B. The graph does not have a "corner" at the height at which the graph becomes linear (neck of flask).

V. Essay

12. (Letter Grade: A - E) In the exam booklet, write an essay on a topic of your choice that is very relevant to the material considered in the course. Your essay should include at least one substantial example and at least one substantial theorem and its proof.

    Good Essays!