MthT 430 Notes Chapter 5c Graphical Binary Expansion Arguments

Binary Expansion Arguments

Consider the following problem:

1. Let $f(x)$ be a function such that

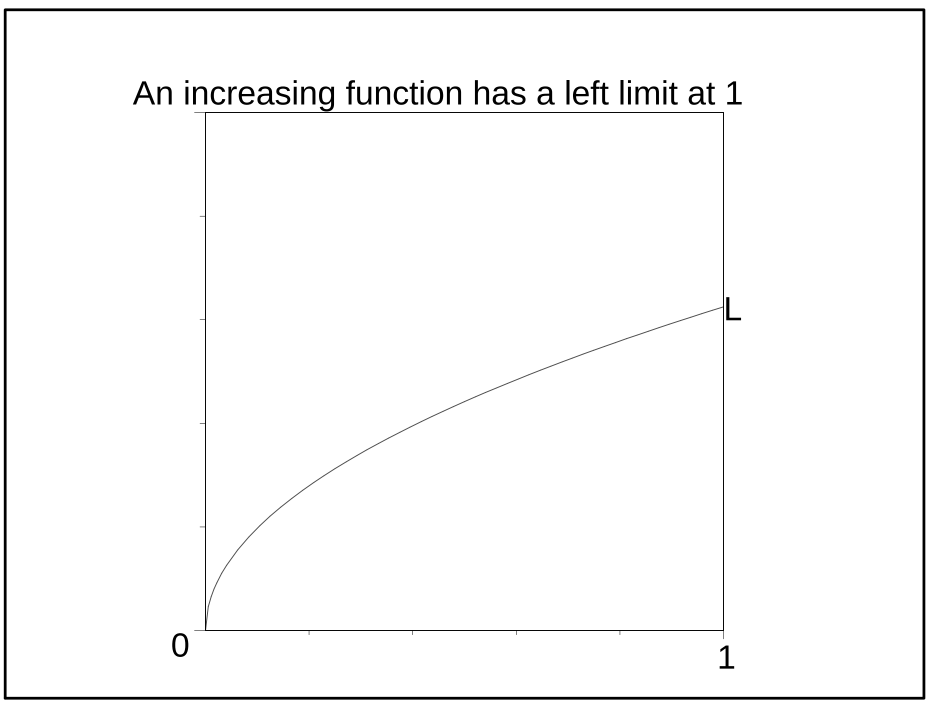
- domain$f = [0, 1)$.
- For all x (in $[0, 1)$), $0 \leq f(x) < 1$.
- The function f is increasing on $[0, 1)$.

Show that there is a number L, $0 \leq L \leq 1$, such that

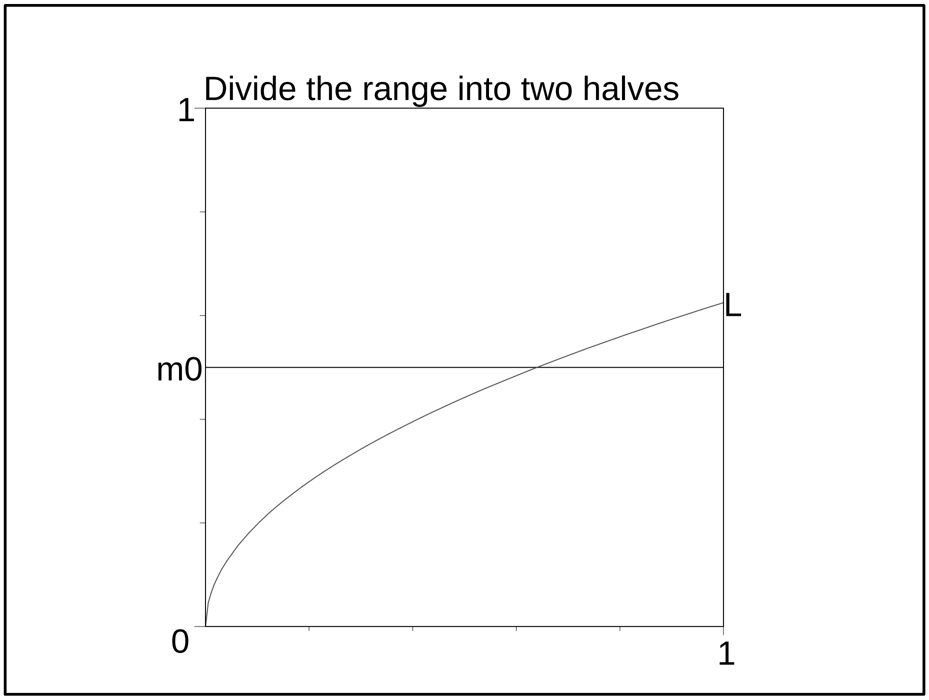
$$\lim_{x \to 1^-} f(x) = L.$$

Hint: Construct a binary expansion for L.

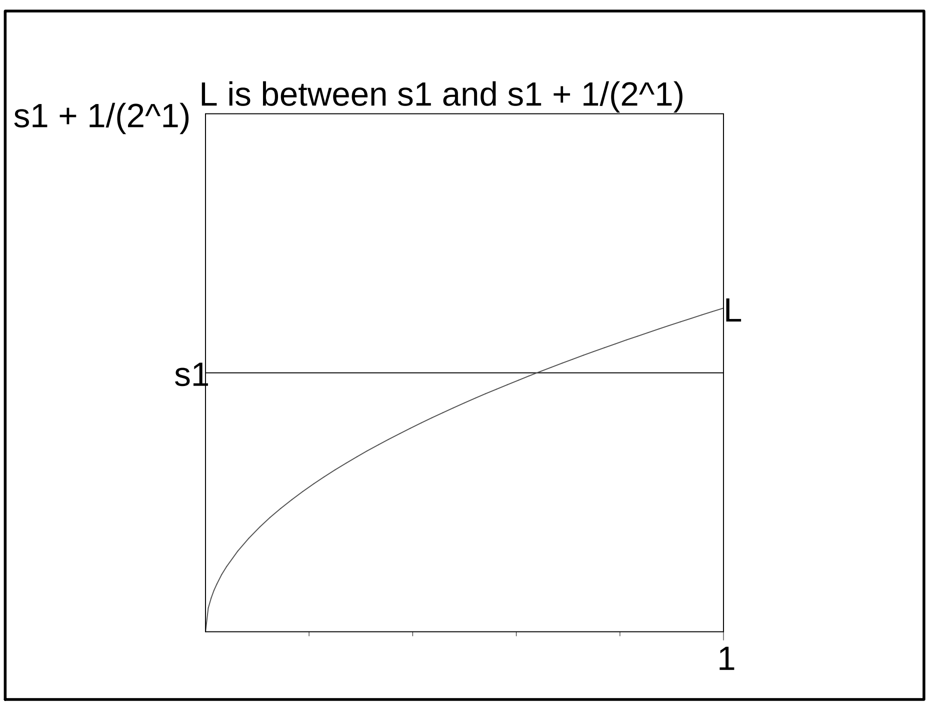
A picture is helpful!



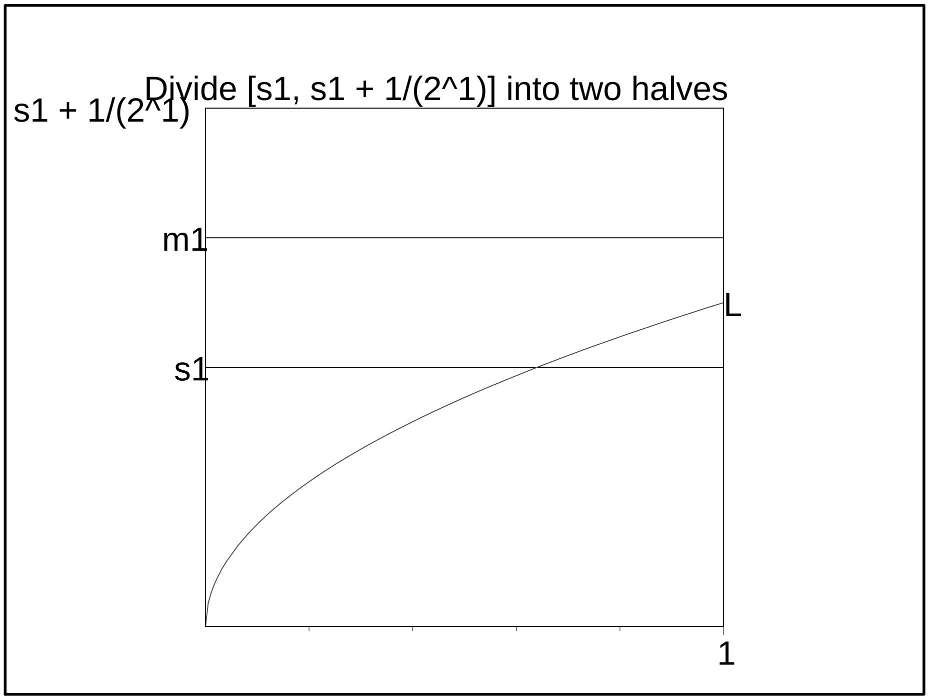
To find the expansion for L, ask the question:
Is there an $x \in [0, 1)$ such that $f(x) \geq \frac{1}{2} = 0._{\text{bin}}1$?



If NO, let $x_1 = 0$, $b_1 = 0$, $s_1 = 0._{\text{bin}}b_1$. If YES, let $x_1 = x$, $b_1 = 1$, $s_1 = 0._{\text{bin}}b_1 = 0._{\text{bin}}1$. In both cases, for $x_1 \le x < 1$, $s_1 \le f(x_1) \le f(x) \le s_1 + \frac{1}{2}$.



Next divide the interval $\left[s_1, s_1 + \frac{1}{2}\right)$ into two parts $\left[0._{\text{bin}}b_1 0, 0._{\text{bin}}b_1 1\right)$ and $\left[0._{\text{bin}}b_1 1, s_1 + \frac{1}{2}\right)$.



Ask the question:

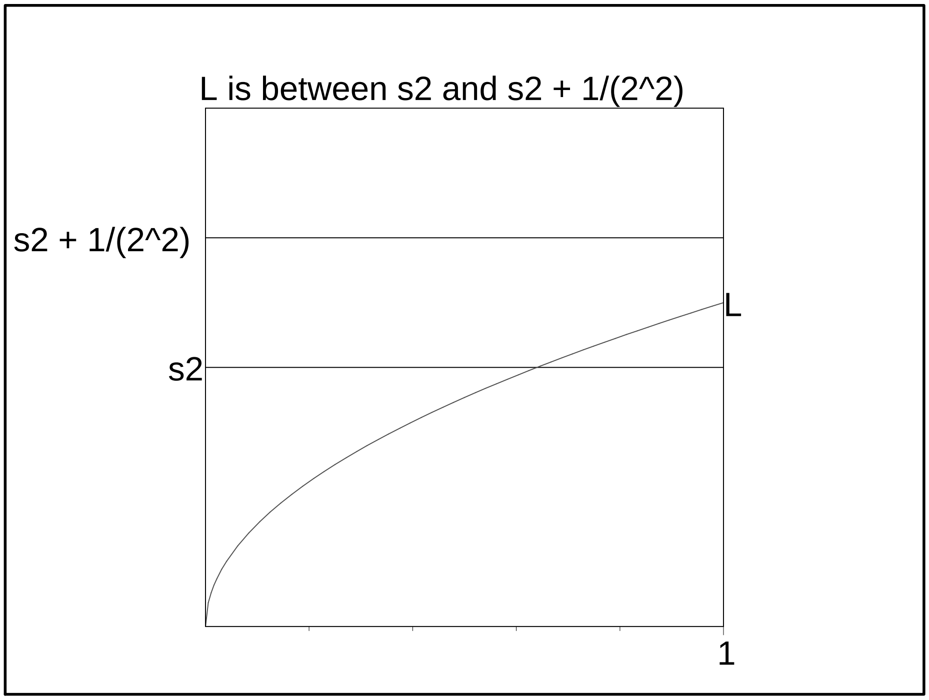
Is there an $x \in [x_1, 1)$ such that $f(x) \geq s_1 + \frac{1}{2^2} = 0._{\text{bin}} b_1 1$?[1]

If NO, let $x_2 = x_1$, $b_2 = 0$, $s_2 = 0._{\text{bin}} b_1 b_2$.[2] If YES, let $x_2 = x$, $b_1 = 1$, $s_2 = 0._{\text{bin}} b_1 b_2 = s_1 + \frac{1}{2^2}$. Then for $x_2 \leq x < 1$, $s_2 \leq f(x_2) \leq f(x) \leq s_2 + \frac{1}{2^2}$.

---

[1] Thinking about this later, I noticed that $s_1 + \frac{1}{2^2}$ is the *midpoint* of the new interval under consideration.

[2] In this case $x_2 = x_1$ and $s_2 = s_1$.



If $x_1, \ldots, x_n$, $b_1, \ldots b_n$, $s_n = 0._{\text{bin}} b_1 \ldots b_n$ have been constructed so that for $x_n \le x < 1$, $s_n \le f(x_n) \le f(x) \le s_n + \frac{1}{2^n}$,

Ask the question: Is there an $x \in [x_n, 1)$ such that $f(x_{n+1}) \ge s_n + \frac{1}{2^{n+1}} = 0._{\text{bin}} b_1 \ldots b_n 1$?

Then let

$$x_{n+1} = \begin{cases} x_n, & \text{NO}, \\ x, & \text{YES}, \end{cases}$$

$$b_{n+1} = \begin{cases} 0, & \text{NO}, \\ 1, & \text{YES}, \end{cases}$$

$$\begin{aligned} s_{n+1} &= 0._{\text{bin}} b_1 b_2 \ldots b_{n+1} \\ &= b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + \ldots + b_{n+1} \cdot 2^{-(n+1)} \end{aligned}$$

$$\text{For } x_{n+1} \le x < 1, \quad s_{n+1} \le f(x_{n+1}) \le f(x) \le s_{n+1} + \frac{1}{2^{n+1}}.$$

Then

$$L = 0.b_1 b_2 \ldots b_n \ldots = \lim_{n \to \infty} s_n$$

since for all x, if $x_n \le x < 1$,

$$s_n \le f(x) \le L \le s_n + \frac{1}{2^n}$$

and

$$0 \le L - f(x) < \frac{1}{2^n}.$$