MthT 430 Projects Chapter 6a Solution

Limits

1. Let $f(x)$ be a function such that

- domain $(f) = [0, 1)$.
- For all x (in $[0, 1)$), $0 \leq f(x) < 1$.
- The function f is increasing on $[0, 1)$.

Show that there is a number L, $0 \leq L \leq 1$, such that

$$\lim_{x \to 1^-} f(x) = L.$$

Hint: Construct a binary expansion for L.

2. Discuss the continuity of the function described on p. 97 and whose graph is sketched in FIGURE 14.

3. Prove: If g is continuous at a, $g(a) \neq 0$, then there is a $\delta > 0$ for which $(a - \delta, a + \delta)$ is contained in the domain of $\dfrac{1}{g}$.

Solution. For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, if $|x - a| < \delta$, then $|g(x) - g(a)| < \epsilon$.

Let $\epsilon = |g(a)|$. Then there is a $\delta > 0$ such that for $|x - a| < \delta$, $|g(x) - g(a)| < |g(a)|$. Thus for $a - \delta < x < a + \delta$, $g(a) - |g(a)| < g(x) < g(a) + |g(a)|$; if $g(a) > 0$, $0 < g(x) < 2g(a)$; if $g(a) < 0$, $2g(a) < g(x) < 0$. In either case, for $a - \delta < x < a + \delta$, $g(x) \neq 0$, and x is in the domain of $1/g$.

Another Solution. For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, if $|x - a| < \delta$, then $|g(x) - g(a)| < \epsilon$.

Let $\epsilon = |g(a)|$. Then there is a $\delta > 0$ such that for $|x - a| < \delta$, $|g(x) - g(a)| < |g(a)|$. Thus for $a - \delta < x < a + \delta$, $|g(x)| = |g(a) + (g(x) - g(a))| \geq |g(a| - |g(x) - g(a)| > 0$. Here we have used the *triangle inequality* in the form $|A \pm B| \geq |A| - |B|$.

Thus, for $a - \delta < x < a + \delta$, $g(x) \neq 0$, and x is in the domain of $1/g$.

Good Variation ... $\epsilon = |g(a)|$ ... If $g(a) > 0$, ... for $a - \delta < x < a + \delta$, $g(x) \in (g(a) - \epsilon, g(a) + \epsilon) = (0, 2g(a))$ and $g(x) \neq 0$. ...