MthT 430 Notes 430 Chap6b Continuity

Continuity at a Point

Definition. *The function f is continuous at a if*

$$\lim_{x \to a} f(x) = f(a).$$

Pay attention to the domain of the function. In particular, if f is continuous at a, then a and all points sufficiently close to a are in the domain of f.

Working Definition. *The function f is continuous at a, if we can make $f(x)$ as close as we like to $f(a)$ by requiring that x be sufficiently close to a*[1].

- (Working JL) The function f is continuous at a, if $f(x) = f(a) + \text{assmallasdesired}$ whenever $x = a + \text{closeenoughto0}$.
- (More Informal) The function f is continuous at a, if $f(x)$ is close to $f(a)$ whenever x is close enough to a.

Thinking of a as fixed, and letting Δx being small (maybe even 0), let

$$\Delta f \equiv f(a + \Delta x) - f(a).$$

To emphasize the dependence of Δf on a and Δx, we somtimes write Δf as $\Delta f(a)$ or $\Delta f(a, \Delta x)$.

Then our working definitions of continuity at a become

- (Working JL). The function f is continuous at a, if we can make $\Delta f(a)$ is assmallasdesired as by requiring that Δx be closeenoughto0.
- (More Informal) The function f is continuous at a, if $\Delta f(a)$ is small whenever Δx is small enough.

$\epsilon - \delta$ **Definition.** *The function f is continuous at a means: For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.*

Variation $\epsilon - \delta$ Definition. *The function f is continuous at a means: For every $\epsilon > 0$, there is some $\delta > 0$ such that, if $|\Delta x| < \delta$, then $|\Delta f(a, \Delta x| < \epsilon$.*

By the fundamental limit theorems, If f and g are two functions continuous at a, then

[1] *Note that we have omitted the phrase "but $\neq a$" but could have included it without changing the meaning.*

- $f + g$ is continuous at a,
- $f \cdot g$ is continuous at a,
- f/g is continuous at a, provided $g(a) \neq 0$.

Compositions

Theorem 2. *If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.*

Proof: $(\epsilon - \delta)$. We must show that: For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, if $|x - a| < \delta$, then $|f(g(x)) - f(g((a))| < \epsilon$.

Fix $\epsilon > 0$. Use the continuity of f at $g(a)$ to find a $\delta_1 > 0$ such that, for all y, if $|y - g(a)| < \delta_1$, then $|f(y) - f(g(a))| < \epsilon$.

Now use the continuity of g at a to to find a $\delta_2 > 0$ such that, for all x, if $|x - a| < \delta_2$, then $|g(x) - g(a)| < \delta_1$.

Then for all x, if $|x - a| < \delta_2$, then $|g(x) - g(a)| < \delta_1$ and $|f(g(x)) - f(g((a))| < \epsilon$.

Continuity on Intervals

A function f defined on an interval $I = (a, b)$ is continuous on I if f is continuous at x for very $x \in (a, b)$.

Continuity of a function on a non open interval requires a modification of the definition of continuity at the included endpoints. A function f defined on a closed interval $I = [a, b]$ is continuous on I if f is continuous at x for very $x \in (a, b)$, right continuous at a – $\lim_{x \to a^+} f(x) = f(a)$ – and left continuous at b – $\lim_{x \to b^-} f(x) = f(b)$. Make the obvious modifications if $I = [a, b)$ or $I = (a, b]$